An Unexpected Appearance
(Or Look Ma, No Rectangles)

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Many papers are a straightforward presentation of the facts,but sometimes there are papers containing a nice surprise. Two of the more exciting types of these gems in mathematics are an object with a counter-intuitive property and the surprise appearance of something from an unrelated topic. In this note we have both of those elements. Our surprise appearance comes from a particular instance in Penney's Game which itself comes from a fun exercise whose base idea is counter-intuitive. This begins with a problem submitted by Walter Penney to the *Journal of Recreational Mathematics* ([6]) in 1969.

> Although in a sequence of coin flips, any given consecutive set of,
> say, three flips is equally likely to be one of the eight possible,
> i.e., HHH, HHT, HTH, HTT, THH, THT, TTH, or TTT, it is rather peculiar
> that one sequence of three is not necessarily equally likely to appear
> *first* as another set of three. This fact can be illustrated by the
> following game: you and your opponent each ante a penny. Each selects
> a pattern of three, and the umpire tosses a coin until one of the two
> patterns appears, awarding the antes to the player who chose that pattern.
> Your opponent picks HHH; you pick HTH. The odds, you will find, are
> in your favor. By how much?

The ensuing game became popular after an appearance in Martin Gardner's *Scientific American* column ([2]).

Penney's Game is a two-player game played via the flipping of a fair coin. Player I picks a sequence of Heads or Tails of length 3 (it can be any agreed-upon length, but in most all literature it is length 3) and makes his choice known. Player II then states her own sequence of length 3. An umpire then tosses the coin until one of the two sequences appears as a consecutive subsequence of the coin flips. The player whose sequence appears first is the winner.

This is one of the members of the collection of *non-transitive games*. Just like Rock-Paper-Scissors regardless of Player I's decision, Player II always has a choice that puts the odds in his/her favor. We show this with the following diagram where the better choice lies at the base of the arrow.
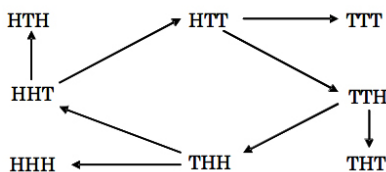
Here are the probabilities associated with the diagram.

Player I's Choice	Player II's Choice	Probability Player II wins
HHH	THH	7/8
HHT	THH	3/4
HTH	HHT	2/3
HTT	HHT	2/3
THH	TTH	2/3
THT	TTH	2/3
TTH	HTT	3/4
TTT	HTT	7/8

Several methods exist to determine the result, including one using of martingales and one due to John H. Conway that uses the binary representations of numbers ([1], [5], [4]).

In this note we focus on one specific situation: Player I has chosen HHT and Player II HTT[1]. This leads us to the surprising appearance.

Assume for our (unfair) coin that p is the probability that the coin land on Tails. Define $x = P(\text{HTT wins})$. We want to find x in terms of p. Now if the opening consists of a string of T's, this has no affect. Similarly, the first H gives neither player an advantage. So assume we begin with $n \geq 0$ T's followed by one H:

$$\text{T} \cdots \text{ TH}.$$

After the H appears in order for Player I to *not lose* we have either

$$\text{T} \cdots \text{ THTT, where Player I wins}$$

or

$$\text{T} \cdots \text{ THTH, where we are in the same situation as T} \cdots \text{ TH}.$$

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[1] We realize that this is not optimal play with a fair coin. The interesting result arose when trying to determine all optimal plays when the coin is not fair.

Thus
$$x = p^2 + p(1-p)x$$
which leads to
$$x = f(p) = \frac{p^2}{1 - p + p^2}.$$
Now the question is, "Can this be a fair game?" What should be the probability of landing on Tails in order for $P(\text{HTT wins}) = P(\text{HHT wins}) = 1/2$? Setting $f(p) = 1/2$ and solving for p yields
$$p = \frac{-1 + \sqrt{5}}{2}.$$
Here is our surprising appearance! This value of p is $1/\Phi$, where Φ is the *Golden Ratio*. Although there is debate about the aesthetic properties of the Golden Ratio ([3]), this result does show us Φ can appear when least expected.

A more thorough investigation of Penney's Game using weighted coins can be found in [7].

References

[1] Collings, Stanley, "Coin Sequence Probabilities and Paradoxes," *Bulletin of the Institute of Mathematics and its Applications*, Vol. 18, November/December 1982, 227 – 232.

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[3] Markowsky, George, "Misconceptions About the Golden Ratio," The College Math Journal, Vol. 23, No 1 (Jan 1992), 2 – 19.

[4] Montgomery, Aaron and Robert W. Vallin, "Penney's Game and Martingales," Proceedings of Recreational Mathematics Colloquium V–G4G (Europe), 2017, 100 – 109.

[5] Nishiyama, Yutaka, "Pattern Matching Probabilities and Paradoxes as a New Variation on Penney's Coin Game," Osaka Keidai Ronshu, Vol. 63 No. 4, November 2012, 269–276.

[6] Penney, Walter, "Penney-Ante," *Journal of Recreational Mathematics*, Vol. 2, 1969, 241.

[7] Vallin, Robert, "Penney's Game with an Unfair Coin," (submitted).