The Foxtrot Half-Empty/ Half-Full Problem

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In early 2006, in the Sunday comic strip "Foxtrot", the precocious ten-year-old, Jason shows the members of his family a cup filled up to the halfway level. He asks each one "Is the cup half-empty or half-full?" His brother and mother say "half-full" and his sister and father say "half-empty". He then laughs and says they are all wrong. "The cup is 5/12 full and 7/12 empty! "


Jason's family is not interested in the answer, but we are. There is a good geometry problem suggested here, namely what is the shape of the cup that will yield that precise ratio?

In the semester starting in January 2006, I was a visiting professor at the University of Georgia teaching in the mathematics department and in the School of Education. I taught a geometry course for 25 secondary education majors, and I challenged them to work on the problem and submit their discussions online before the next class.

Since the inspiration for the problem was a comic strip, most students started with the two-dimensional problem. The cup was modeled as an isosceles trapezoid with height h, top edge of width a, and bottom edge of width b, shorter than a. The size of the cup was given by the area, with the whole cup having area $h(a + b)/2$. The width of the line halfway up at height $h/2$ was the average $(a+b)/2$.
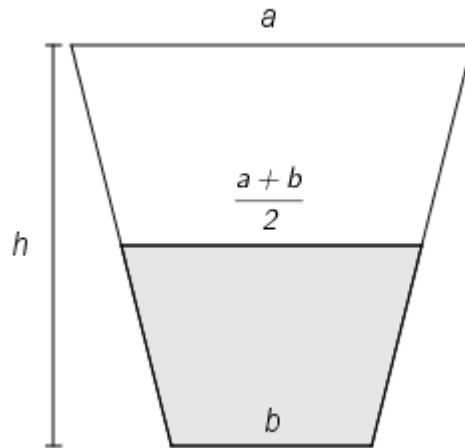
So the upper region was a trapezoid with height $h/2$, with top edge of width a and bottom edge of width $(a+b)/2$, averaging $[a + (a+b)/2]/2 = (3a + b)/4$ for a total area of $h(3a + b)/8$. Similarly, the area of the lower region was $(h/2)[(a+b)/2 + b]/2 = h(a + 3b)/8$. The ratio of the area of the upper part to the area of the lower part was then $(3a+b)/(a + 3b)$, independent of h. As a check, if $b = 0$ and the cup is a two-dimensional Dixie cup, then the ratio of area of upper part to area of lower part is $3/1$, which checks since the lower part is an isosceles triangle and the upper part can be decomposed into three isosceles triangles of the same size.
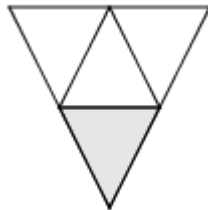
If b is positive, then we can divide the top and bottom of the ratio by b to get $(3a+b)/(a + 3b) = [3(a/b) + 1]/((a/b) + 3)$ so the ratio of upper to lower areas depends only on the ratio $r = a/b$, namely $(3r + 1)/(r + 3)$. To find the unique ratio r giving Jason's ratio $(7/12)/(5/12) = (7/5)$ we need to solve $(3r+1)/(r+3) = 7/5$ so $5(3r+1) = 7(r+3)$ and $15r +5 = 7r + 21$ so $8r = 16$ and $r = 2$. The top edge is twice the width of the bottom edge, which looks like a very good approximation of the ratio of top edge to bottom edge of the cup in the comic strip.

Even though the analytic geometry argument based on the work of several students was quite convincing, even more convincing was a one-diagram geometric "proof by picture". A trapezoid with top edge of width 4 and bottom edge of width 2 can be divided into 12 congruent isosceles triangles with base of width 1, with 7 triangles in the top part and 5 in the bottom part! One student wrote the single word "Walla!" on the bottom of this diagram and when I asked what he meant by that, he said "You know, the French word, Voila!"
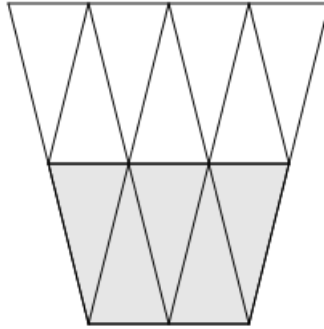
Although this kind of proof is quite convincing when the ratio of the top width to the bottom width is a rational number, there is no good diagram of the same sort when that number is irrational.

As it happens, there were three PhD candidates observing the class, and one of them, my G4G13 co-presenter Tom Cooper, now a full professor at the University of North Georgia. He interpreted the Foxtrot problem as three-dimensional and that led up to a more complicated situation. Instead of a trapezoid, he considered the frustum of a right circular cone of height h with radius a for the top circle and radius b for the bottom disc. In the class we had already showed that the volume of such a cup is $(\pi/3)h(a^2 + ab + b^2)$. A horizontal slice at height $h/2$ would have radius $(a+b)/2$, so the volume above that slice would be $(\pi/3)(h/2)[a^2 + a(a+b)/2 + ((a+b)/2)^2]$ and the bottom below the slice would have volume $(\pi/3)(h/2)[((a+b)/2)^2 + ((a+b)/2)b + b^2]$.

As in the two-dimensional case, the ratio of the volume of the upper region to the volume of the lower region is independent of h and only depends on the ratio $r = a/b$ of a to b. The formula for this ratio is more complicated in the three-dimensional case, involving quadratic factors:

$$\frac{a^2 + a \cdot \dfrac{a + b}{2} + \left(\dfrac{a + b}{2}\right)^2}{\left(\dfrac{a + b}{2}\right)^2 + \dfrac{a + b}{2} \cdot b + b^2} = \frac{r^2 + \dfrac{r(r + 1)}{2} + \dfrac{(1 + r)^2}{4}}{\dfrac{(1 + r)^2}{4} + \dfrac{r + 1}{2} + 1}$$

If $r = a/b = 2$, then the ratio of the top part to the bottom part is $(4 + 3 + 9/4)/(9/4 + 3/2 + 1) = (37/4)/(19/4) = 37/19$, not such a simple calculation as in the two-dimensional case. So if Jason lived in a three-dimensional world, he could have said, "The cup is 19/56 full and 37/56 empty!"

That solves the problem for a surface of revolution of an isosceles trapezoid, but Jason's planar cup has different analogs in the third dimension. For example, a truncated inverted "Egyptian" pyramid with square top of side length a and square bottom of side length b. Analogous to the volume of the truncated cone, the truncated pyramid has volume $V = (1/3)h(a^2 + ab + b^2)$.

For the case of $a = 4$ and $b = 2$, there is an analogous demonstration similar to the "Viola" case using pyramids and tetrahedra. The halfway slice is a square with side length 3.
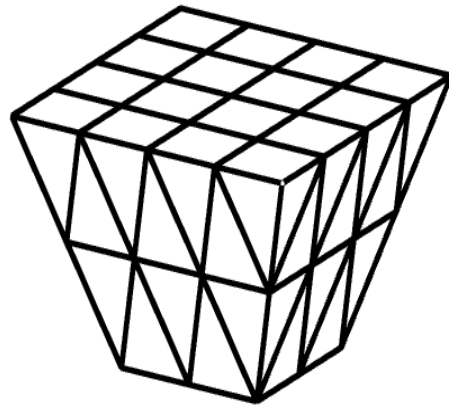
We can divide the lower region into four square pyramids pointing up and nine congruent pyramids pointing down (producing 13 congruent pyramids for G4G13). Those sets of pyramids correspond to the terms a^2 and b^2 in the formula for the volume of a frustum of a square pyramid.
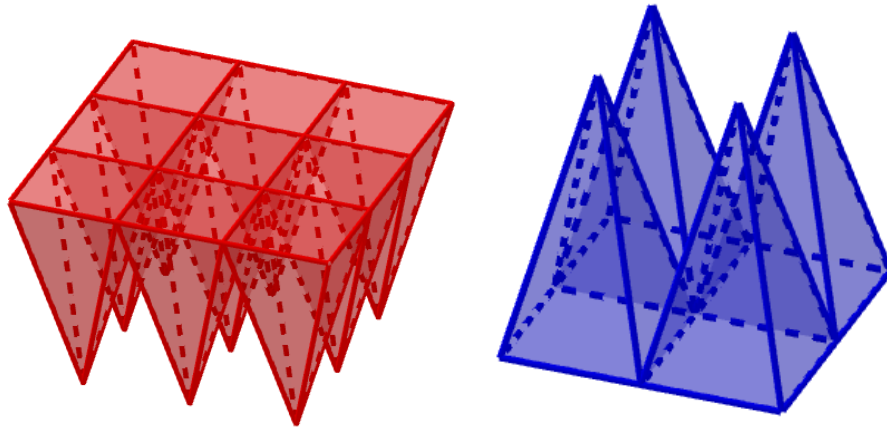
In addition, there are 12 tetrahedra corresponding the ab term in the formula for the volume of a frustum.
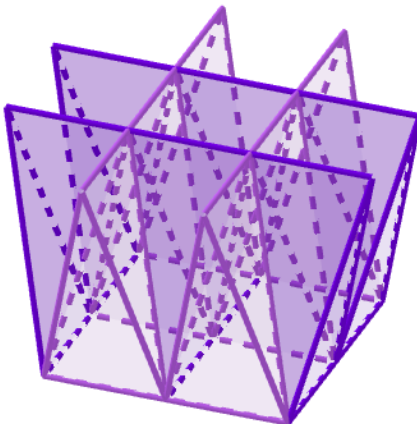
The truncated cones and pyramids are only two kinds of generalizations of Jason's two-dimensional cup. There are other considerations in dimension three, or even higher! We would like to thank Jason and his creator Bill Amend for inspiring our geometric excursion.