

Towers and Dragons: An Unexpected Connection

Introduction

The Tower of Hanoi is a puzzle so well known that it hardly needs an introduction. This paper connects efficient solutions to the Tower of Hanoi and the creases formed when a strip of paper is repeatedly folded in one direction, known as the Dragon Fold. What I've written here is a concise explanation of my "Towers and Dragons" lesson plan that was the winning submission for the 2020 Rosenthal Prize; for a fuller treatment see momath.org/rosenthal-prize/.

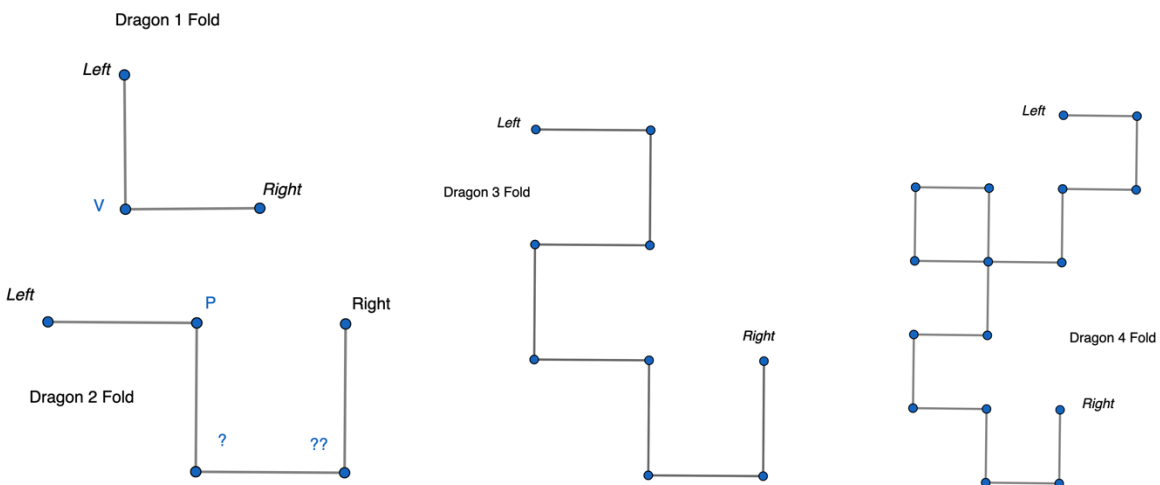
The Tower of Hanoi

This puzzle is so well-known that I'll give a very concise description here. n discs are labeled 1, 2, 3, ..., n , which are decreasing in size as n increases. The discs are stacked on one of three pegs, with the discs decreasing in size as you ascend. The goal of the puzzle is to move the entire stack from one peg to another. The constraints are that you can only move one disc at a time, and that smaller discs can be stacked on top of larger discs but not vice-versa.

The minimum number of moves required to complete the puzzle is $2^n - 1$, which follows by a simple inductive argument. Moreover, the sequence of moves to achieve this minimum follows a symmetric/fractal like pattern. For example, with 4 discs, the sequence of moves is 434243414342434.

The Dragon Fold

Take a strip of paper and fold the left edge to the right edge. Repeat this n times, always folding from left to right. This is a dragon fold, so called because when we unfold the strip the result is a dragon curve:



The first connection to the Tower of Hanoi is obvious: An n th-stage Dragon Fold has $2^n - 1$ creases.

But the connection to Hanoi is deeper still. Suppose we label a crease formed by the first fold 1, then the creases formed by the second fold 2, etc. Read left to right, at the 4th iteration the creases are numbered: 434243414342434! So we can use the creases of the Dragon Fold to solve the Tower of Hanoi puzzle.

An extension: Peaks and Valleys

Another way to label the creases in a Dragon Fold are to indicate a 'V' whenever that crease is a Valley Fold, and a 'P' when the result is a Peak. The first iteration of the Dragon Fold gives the sequence V; the second gives PVV (in the figure above, ? and ?? are both Vs). The 3rd and 4th iterations give:

PPV V PVV

PPVPPVV V PPVPPVV

These, of course, are not symmetric in the same way as the sequences of numbers given earlier. How can we generate each successive iteration of sequences?

First, note that the middle symbol in any iteration is V.

Next, notice how the right side of the sequence for a given iteration is simply the previous iteration.

To derive the left side of the new iteration, visualize unfolding the dragon. What happens to all of the peaks and valleys as we unfold from right to left?!