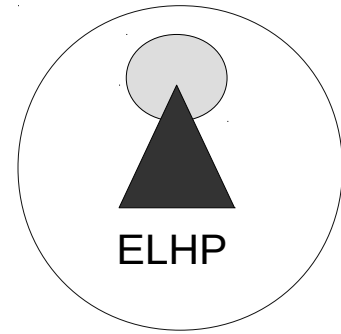


Mathematics and Art: The ELHP

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Some people like to hear about mathematics being used to address real-life problems. I am going to claim that the problem I describe in this article is real-life because it arises from a conversation between two non-mathematicians.

Specifically, one of my sisters-in-law did an art degree, and as part of a project she did for this she visited Sardinia to interview the sculptor Pinuccio Sciola. At least some of his works are quite large, by which I mean maybe 3m high or more, based on things I see on the web. During their conversation, he said something about wanting to install one of his sculptures on a named mountain somewhere in the Catania area, for the benefit of the residents. I don't know his exact words, but my sister-in-law found this remarkable enough that she reported it to me and other members of the family.

It may be important to note that my sister-in-law lives in a small town near Catania, and this may be what prompted him to say this. It seems entirely likely that he had not spent any real time looking into this idea. Indeed, I am told that when he later visited Catania he immediately realised that his idea was probably unrealistic.

Let's use maths, and some other disciplines, to consider his idea, pretending for the sake of discussion that he or someone else really does want to go ahead with it. You may notice that even though Sciola named the mountain, I have chosen not to do so. I shall do this later.

For starters, let's have a look at the mountain. I took this photograph from just outside Catania airport.



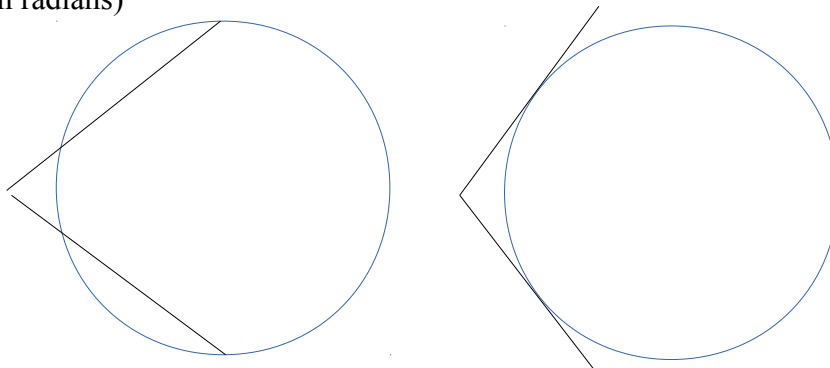
Catania itself is mostly nearer to the mountain than the airport is, but if we're going to build a sculpture or statue on a mountain to be seen from a city with an international airport, we might as well make it so that people arriving from overseas can appreciate the statue as soon as they arrive. Apart from anything else, when you're in town, buildings are often between you and the mountain.

I don't know that Sciola said where on the mountain he wanted to place a sculpture, but it seems as though if you're going to do this at all you might as well put the sculpture right on top of the mountain.

Our first discipline might be geography: how far is the peak of this mountain from Catania airport? There is a "measure distance" feature on Google Maps, and drawing a line from where I took the photo to a reasonable guess at where the peak is, I get 32km. There's also an issue of vertical separation, since the mountain is 3329m tall. There are plenty of higher mountains in the world, but that's not bad. (From my father-in-law's house to the peak is about 26km horizontally.) The airport is very near the coast, and my eyes aren't that far above the ground. Let's pretend the photograph was taken at exactly sea level. We're not going to be pretending anything we do has more than about 1 significant figure anyway. Using Pythagoras we get straight line distance as $\sqrt{3329^2+32000^2}=32173\text{m}$. Let's just say 32km.

Since we can see the mountain, we could at least in principle see things on it. But how big would they need to be? We're probably asking how large a solid angle in steradians the sculpture needs to subtend, or what its angular diameter needs to be in degrees or radians. If we assume the sculpture to be approximately spherical, we don't need to worry about foreshortening caused by looking up at it either. Since we know we can see very distant stars in the night sky, and they have extremely small angular diameters, it's clear that there isn't really a minimum size our sculpture could be to be visible at all: if it gives off enough light we'll be able to see it no matter how small it is.

Here we see that the angle subtended at the eye by the diameter of a sphere (perpendicular to line of sight) is smaller than the angle subtended at the eye by the sphere as a whole. Clearly in this example both angles are quite large and the difference between them is substantial. With much smaller angles, as we will be discussing, the difference is much smaller since x , $\sin x$ and $\tan x$ are all very similar for small x . (With x in radians)



When I calculate the sizes of statues at given distances I will just use $2\pi r(\text{angle}/360)$ (with the angle in degrees). For the very small angles we will be talking about, this is good enough. I am ignoring the fact that the distance to the centre of the statue increases as the statue gets larger. Taking the height of the mountain into account did not make much difference and for our purposes we can for now ignore the extra distance induced by changes in statue size. And the fact that the Earth is not a perfect sphere.

But would a sculptor be satisfied with a sculpture that is for all practical purposes a single pixel? I'm not an artist, so don't really know, but it seems unsatisfactory. I can see that an artist might choose the colour of a single pixel with great care. The intensity and colour could even vary with time if we wanted something more than a static dot. However, let us suppose that this is not what Sciola had in mind.

At this point we need to know something about human physiology. About how big should the smallest details of the sculpture be for people to be able to see those? It appears (see e.g. <http://www.bbc.com/future/story/20150727-what-are-the-limits-of-human-vision>) that the finest pattern the human eye can distinguish is about 120 lines per degree. So single dots are half a minute of arc. At 32km, half a minute of arc works out to be about 4.7m. Let's say 5m. We could start to think in terms of pixel art with 5m pixels but perhaps this approach is wrong.

We could instead consider existing statues or other items which are viewed from some distance away, and where more than their mere existence can be perceived, and ask ourselves what their angular diameters are when viewed from those distances. If these statues or other objects are approximately spherical, so much the better. We can of course consider things such as the Angel of the North, Christ the Redeemer and so on but the Moon seems to be a very good candidate. It's approximately spherical, its angular diameter is about half a degree, and "about the same size as the Moon" would not be an outrageous starting point for what we might want a statue to be. If we want some kind of "moon unit" like this to use in comparisons, we could call it the Zappa. Half a degree, at 32173m, means a diameter of 281m.

What should our statue be of? One might think that the patron saint of Catania, Saint Agatha, would be a candidate, or the elephant, the symbol of Catania. The problem with these is that the top of the mountain can be seen from other, closer, communities who might feel that this was unreasonable. And Saint Agatha, as usually depicted, is not approximately spherical.

I submit, then, that a solution which should be acceptable to all is to build a statue of my stuffed hedgehog, Herisson. (Photo by Adam Atkinson)



Herisson is round enough that I think we can treat him or her as approximately spherical.

And so we have the Extremely Large Herisson Project: we wish to erect a statue of Herisson on top of a mountain 3.3km tall, 32km away, so that it appears to be about the same size as the moon. Let's say that we want Herisson's largest dimension to subtend the same angle that the Moon does.

As seen above, this works out to be about 281m. This is of course a lot. However, looking for large statues we find that the Spring Temple Buddha in China is 128m tall, on a 25m base. Our proposal is not fantastically larger than the largest statue in the world. Alternatively, we could settle for the Extremely Large Herisson being a little less than one Zappa in size, perhaps.

There are non-mathematical considerations, however. Merely doing calculations with similar triangles ignores the problem of haze. When in Catania recently trying to take a photograph of the mountain, 5 days out of 7 it was behind haze, fog, cloud or similar. Being able to see anything at all at 32km, especially when it is above common altitudes at which clouds are found, is problematic.

One is also driven to wonder how planning permission for this project would work. On Wikipedia we find this image, (By Skyluke - Own work, Public Domain, <https://commons.wikimedia.org/w/index.php?curid=1961715>) which purports to show the town boundaries in the Catania area.



The mountain peak is the point in the top right where multiple sectors meet. It might be necessary to obtain planning permission from all the towns whose areas meet at the summit. Might it be necessary to get two lots of planning permission from the town whose area meets the summit twice? Of course, since much of the mountain is a national park, there could be further complications. I may ask at least one of the town councils about this project at some point. I have in fact written to one town council, that of Nicolosi, asking about this. At the time of writing, in late June 2018, I have had no reply.

Building a 281m statue on top of any 3329m mountain would be a challenge, but in this case there are extra difficulties: The mountain under discussion is in fact Mount Etna, one of the most active volcanoes in the world. Google can find plenty of images of spectacular events on and around Etna.

One experiment we can do is to view normal-sized Herisson at suitable distances to see if one Zappa does indeed seem about right, and also try to see if when his or her angular diameter is half a minute he or she is indeed visible at all. (He or she is white, so against many backgrounds might stand out enough to be seen.) A very long tape measure would of course be needed.



Here I am on a field trip to Catania (photo: Annamaria Cucinotta), holding up a copy of Chalkdust magazine with Etna in the background. Ideally we should have made sure the distance to the camera and the size of the magazine were such that it looks the same size as the Moon would on the mountain behind me. This photo serves to illustrate a way we could decide what a decent size of statue would be. A stepladder would clearly have made it possible to hold the magazine so that it seemed to be on top of Etna, but one was not immediately to hand.



On a later field trip it seemed useful to examine the proposed construction site itself. On the left I am in front of the main crater with a normal-sized copy of Chalkdust. The photo on the right has some people in it to give a sense of the scale of the place. (Both photos by Gunther Schmidl) The main thing I learned on this trip was that I never want to do it again. Making it all the way up to the crater is quite hard. Transporting construction materials for the sculpture would be a major undertaking.

But why stop at Catania? The city of Messina is about 70km away. And if we calculate the distance to the horizon from the top of Etna we discover that it's about 206km. 160km gets us to Palermo. 203km gets us to Catanzaro in southern Italy. Most of Sicily is within 206km of the peak, but we can't quite get to Trapani or Marsala on the west coast of Sicily. However, the sculpture would be so big by this point that its feet on the mountain would be invisible but some of the sculpture could still be seen. Still, I think we have to insist that the feet be visible: if we make a tall thin statue big enough of course the top can be seen from almost half the planet, and some part of a large enough spherical statue can be seen from almost anywhere on the planet (though the statue may need to be larger than the planet itself). Also, if your eyes are higher than sea level you can see some extra range from that. One photograph I have seen online of Etna from Catanzaro was taken from a hotel balcony. It seems possible that from a tall enough building in Trapani or Marsala, the top of Mount Etna might just be visible if there's nothing in the way.

Note that I am completely ignoring refraction here. In real life, refraction could make a difference, and as the wonderful article by David E.H. Jones (a.k.a. Daedalus) in "New Scientist" <https://bit.ly/2lzs wd8> observes, if we replaced the Earth's atmosphere with sulphur dioxide or reduced the Earth's radius we could in principle see as far as we wanted. But both of these options are well outside the scope of the ELHP.

Putting 206km instead of 32km into our formulae, we get "pixels" of about 30m and statue size 1798m. At this point we have a Prodigiously Large Herisson Project, and it seems quite impractical. And there's the question of which way he or she should be facing. Which cities get to gaze upon Herisson's gigantic backside? Gunther Schmidl (personal communication) suggests making the statue rotate about a vertical axis. This is clearly the way to go.