A Recipe for a ‘bola Honeycombs

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Abstract:

A hexagonal grid and simple integer addition can be used to generate a set of coordinates which all fall on a three-dimensional surface called a paraboloid. The grid generates the vertices of affine hexagonal facets which bound an infinite polyhedron that I have dubbed the "Parabolahedron" (hence 'bola in the title). The entire parabolahedron is completely determined by the choice of four "seed numbers" from which the entire polyhedron is derived. An infinite variety of these parabolahedra may be generated by choosing different "seeds".

Once completed, a grid can be used directly for modeling a paraboloid by stacking anything from coins to Honeycomb cereal in stacks corresponding to the values in the grid. The result is a paraboloid shaped "bowl".

Since the process of generating the honeycomb grid requires only integer addition, it is a suitable puzzle for students in early elementary grades. Any of several pre-calculated grids could be provided to allow even younger students who are just learning to count to generate variants of the "bowl" shape by stacking simple objects such as coins. At the same time, the geometric progression of the grid can be used in more advanced studies of topics including averages, slope, volume, exponential growth, symmetries, conic sections, quadric surfaces, vector addition and affine transformations.

Introduction:

A hexagonal grid is the basis for a simple, yet intriguing puzzle presented here in two parts. The first part is the 2D puzzle itself which is reminiscent of Pascal's Triangle both in its simplicity as well as the hidden gems it reveals. The second part takes advantage of the printed puzzle to make some actual 3D models of the shape. In closing, I’ll offer some ideas for use of this puzzle in various classroom environments.

The Honeycomb Grid:

The grid consists of any number of identical hexagonal cells laid out like a honeycomb as shown in Figure 1. The hexagons need not be regular. That is, it is not necessary for them to all have all edges the same length or all angles identical, but it is necessary for them to have all three pairs of opposite sides parallel to the diagonal between each of them. The following examples uses regular hexagons to introduce the concepts. The vertices of the grid where three hexagons join are indicated in red. The centers of each cell (called faces) are indicated in green and the edges of each cell are bounded by a pair of lines which will occasionally be designated “slope”.

An arbitrary vertex is chosen. It’s shown in bright red in Figure 1 surrounded by three edges highlighted in yellow. The grid is populated by selecting any numbers for the chosen vertex and the three adjoining yellow edges, then calculating values for the remaining vertices, faces and edges according to the following rules.

1) The value on any edge is copied to the opposite parallel edge as well as each side of the center between the green and red points as shown in Figures 2a, 2b and 3.
2) The values of two edges with a common vertex are added to produce the remaining edges, which is then copied to the two remaining internal positions.
3) The initial red cell is added to the adjoining edge values to produce the values of the adjacent vertices which are in turn added to their adjacent edge to produce the 4th and 5th vertex.
4) Either the 4th or 5th vertex can be added to its adjacent edge to derive the final vertex, which is the one opposite the first one.
5) The value of the face, indicated in green can be determined in any of several ways, all of which should produce the same value. It is the sum of the vertex closest to the initial red vertex and the "slope" value between that vertex and the middle. It is also the value of vertex farthest from the initial red minus the "slope" between it and the middle. It is also the average of every pair of opposite vertices for that hexagon.
6) If the various methods for calculating the green value in a cell don't all agree, then stop and check your earlier math before continuing to the next cell. Figure 4 shows an example of a completed cell.
7) The new edge value determined in step 2 can now be copied to all applicable positions in the grid as described in step 1 and then repeat the process for any hexagon having a value for its vertex and two adjacent edges.

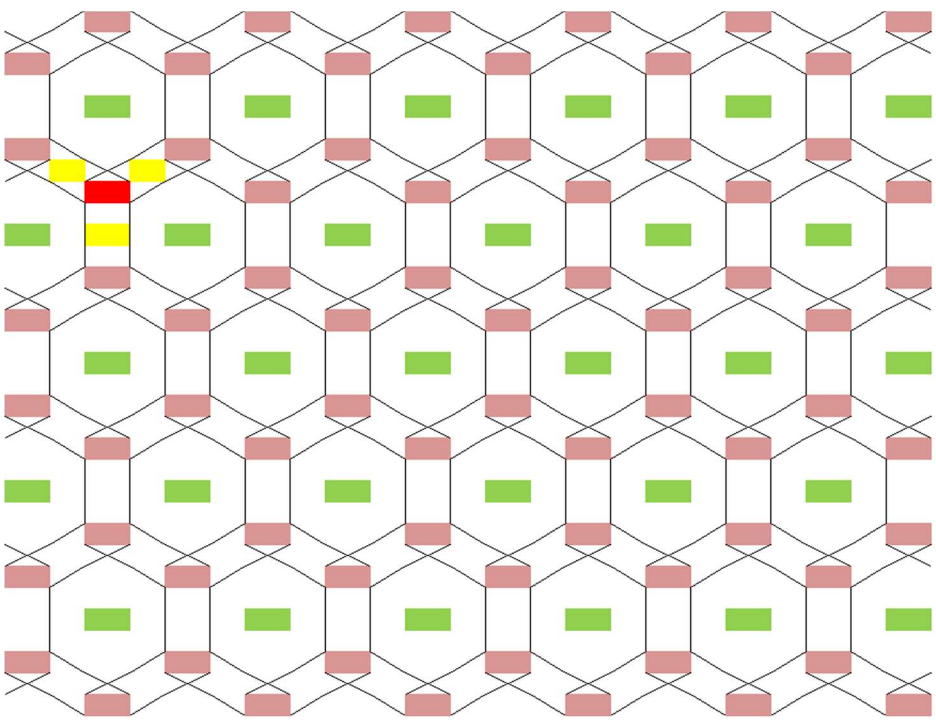

Figure 1: Blank honeycomb grid with an arbitrary starting point highlighted.

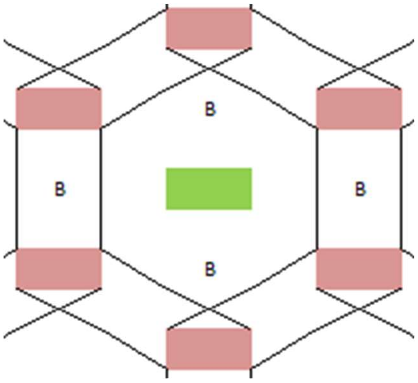

Figure 2a: Slope “B” is copied horizontally.

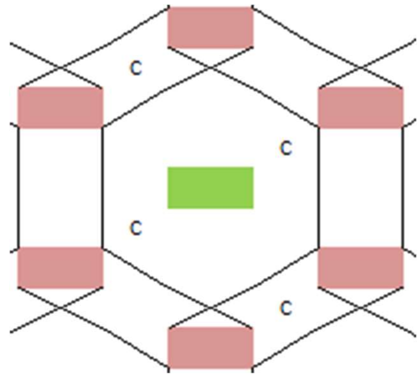

Figure 2b: Slope “C” is copied diagonally.

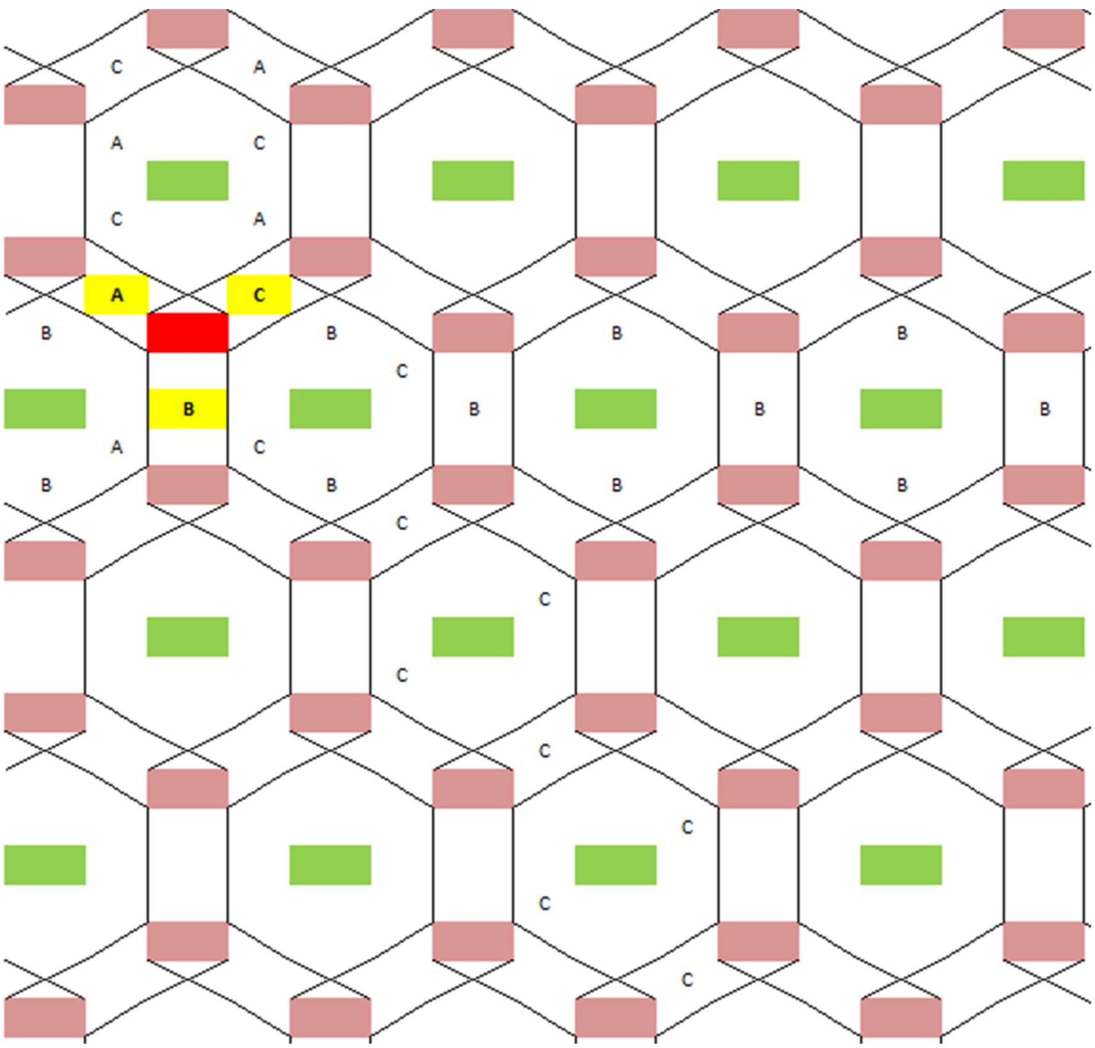

Figure 3: Initial edge values of A, B and C can be copied to the rest of the grid as shown. The actual puzzle would use numbers, not letters in these positions.

Figure 4: Relationship between vertices, edges and middle of each cell.

Here's a step-by step example using the cell on the right side of Figure 4:

1) Values of 2 and 3 are copied from the two yellow edges to opposite edges and internal slopes.
2) 2 + 3 = 5 which completes the edges and internal slope values.
3) 0 + 2 = 2 and 0 + 3 = 3, then 3 + 5 = 8 and 2 + 5 = 7
4) Either 8 + 2 = 10 or 7 + 3 = 10 which is the last vertex of this cell.
5) The green center is 0 + 5 = 5 or 10 – 5 = 5 or average(0,10) or average(3,7) or average(2,8).
6) The cell on the left is solved similarly. The other cells around them are shown partly solved.

<u>What's the point?</u>

Now that we have our grid, we can use the numbers in red to get a list of three dimensional coordinates. The numbers in green and white areas will come in handy later too. Those in white, as you may have already realized, have to do with the slope of a facet. Those in green are important coordinates too, but they don't lie on the surface of the paraboloid. They're useful for calculating volumes too as we'll see.

For now, let's just determine the list of 3D coordinates. It was mentioned earlier that the hexagons don't need to be regular. They just need opposite edges and their diagonals to be parallel. With that in mind, we can use any unit for measuring our grid in the X and Y directions. I prefer to start with 0 in the bright red vertex, then I can simply count hexagons vertically or horizontally. The hexagon in Figure 4 is divided into smaller rectangles. They provide an easy way to count off the X and Y coordinates. Since we won't be concerned with the white slopes as coordinates, we can skip them and just count every other column for the X coordinate and every other row for the Y coordinate, and best of all, we can use the value in the red cells directly as a Z coordinate. Again, the units don't matter. Our X, Y and Z coordinates don't even need the same units as each other.

With this basis for measuring our set of coordinates, the vertices shown in Figure 4, listed from left to right and top to bottom are: (-1, 1, 1), (1, 1, 3), (-2, 0, 4), (0, 0, 0), (2, 0, 8), (-2, -2, 6), (0, -2, 2), (2, -2, 10), (-1, -3, 5), (1, -3, 7). As a side note, it's interesting to point out that, just as 3 points define a circle and 5 points define a conic section (e.g. ellipse or hyperbola), 9 points are needed to define a quadric surface like our paraboloid, and we have calculated 10 points so far. Calculating the formula for our paraboloid is beyond the scope of this article, but it's interesting to note that we already have enough information to do so.

<u>Honeycombs… They're not just for breakfast any more.</u>

But wait, since the dimensions are all independent and can use different units, we can finally start to develop our simple 3D model.

Let's simply use the grid directly without counting off X or Y coordinates, and then we can decide that a unit in the Z direction (perpendicular to the grid) is whatever we have handy. If you have a few dollars in pennies available, then define one unit in the Z direction as the thickness of one penny, then make stacks of pennies right on top of each red vertex. The number in the vertex represents the number of pennies in the stack. If you started with big numbers, your bowl will get expensive. If you started with small numbers, like 1's and 0's, the thickness of a penny won't add up to a very deep bowl, and that's where the Honeycombs come in. Honeycomb cereal is about 4 to 5 times as thick as a penny, so they can be stacked in similar fashion and the bowl will grow more quickly so you'll finally start to see your "bola" Honeycombs. Of course, you're free to experiment with this recipe and substitute just about anything you'd like for the stacks of Honeycombs. The result will still be points on a paraboloid.

Figure 5: Piles of Honeycomb cereal stacked on the hexagonal grid begin to curve upward and form the promised 'bola Honeycombs. Pennies or other items can be substituted to form the stacks. The cereal was laced onto Angel hair pasta to keep them from falling over. Pennies are easier to manage.

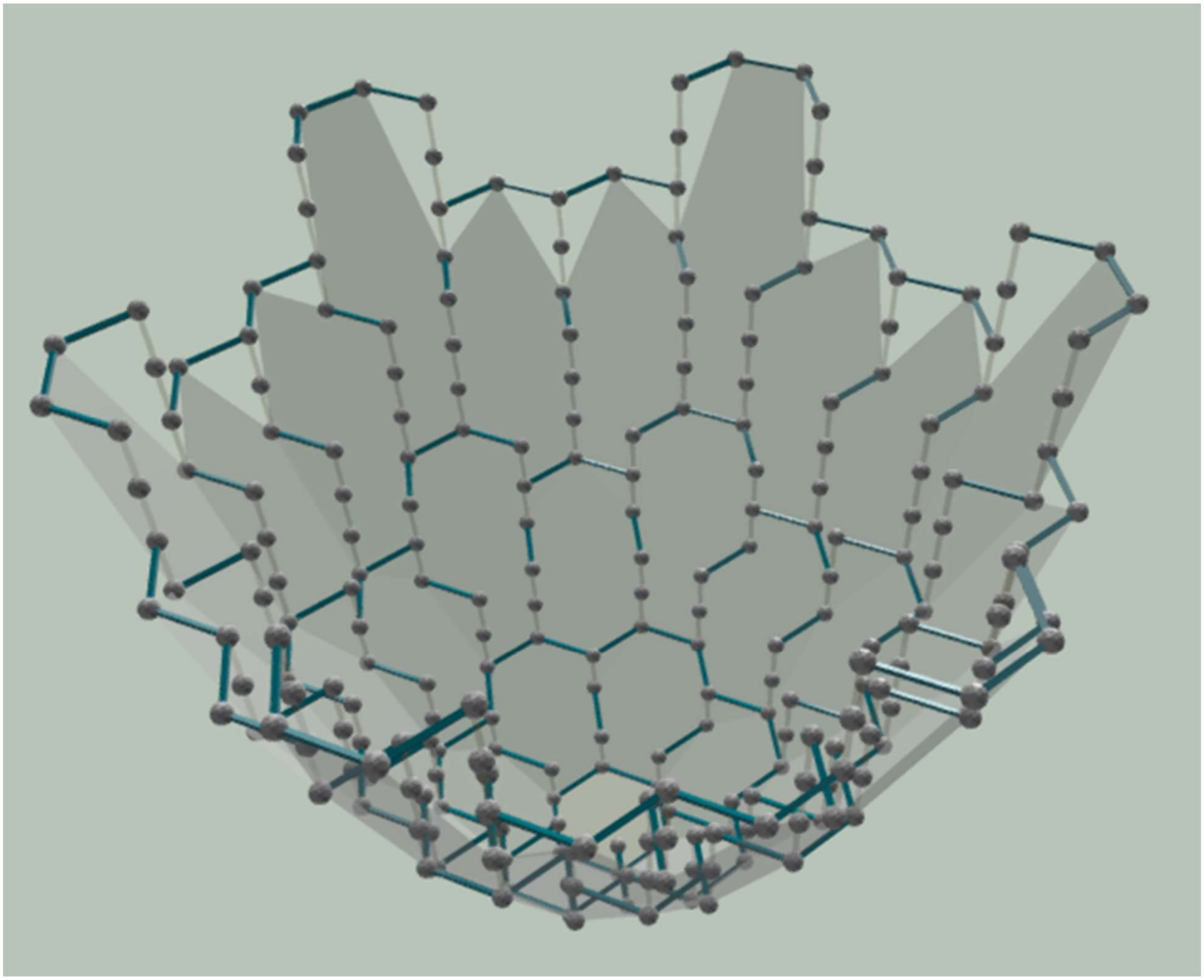

Figure 6: A parabolahedron made by stacking zometool balls and struts. This interactive 3D model is available on Sketchfab at https://skfb.ly/6wXPF.

Consider the alternatives:

At this point, we'll look at the outcomes from choosing various initial values. This will not be an exhaustive study, but it will point out some key characteristics to notice when experimenting with the grid.

Ultimately, the grid is the basis for modeling our paraboloid, or more specifically, a subset of a paraboloid which I've dubbed a Parabolahedron, so let's continue with a few simple definitions.

1) A parabola is a conic section. It's the two-dimensional shape that results from slicing a cone parallel to its surface.
2) If you spin a parabola around its axis of symmetry, you get a paraboloid. It's a three-dimensional shape that you might recognize as a reflector for a lamp. If you slice it perpendicular to its axis, you'll find a circle.

3) Squashing a paraboloid so that the circular cross section becomes an ellipse is still a paraboloid. Unsurprisingly, it's called an elliptical paraboloid.
4) Any version of a paraboloid is a smooth continuous three-dimensional curve called a quadric surface. All the points we've generated with our grid will fall on that surface, but they will not completely fill the surface.

It turns out that each hexagon generates a set of 6 points which, even after being projected into a 3rd dimension, remain coplanar. In other words, each original hexagonal cell in our grid corresponds to an affine hexagonal facet of the parabolahedron.

The initial point has been chosen well off center in these examples to feature the extended growth of the red and green coordinates. Symmetries can be used to enlarge the grid based on calculating and then replicating a subset of the coordinates. Regardless of the first vertex we choose, the initial choice of slopes will determine the orientation of the parabolahedron with respect to the plane of our grid. Let's assume for now that the initial vertex is at 0. Several possibilities are described in the following figures.

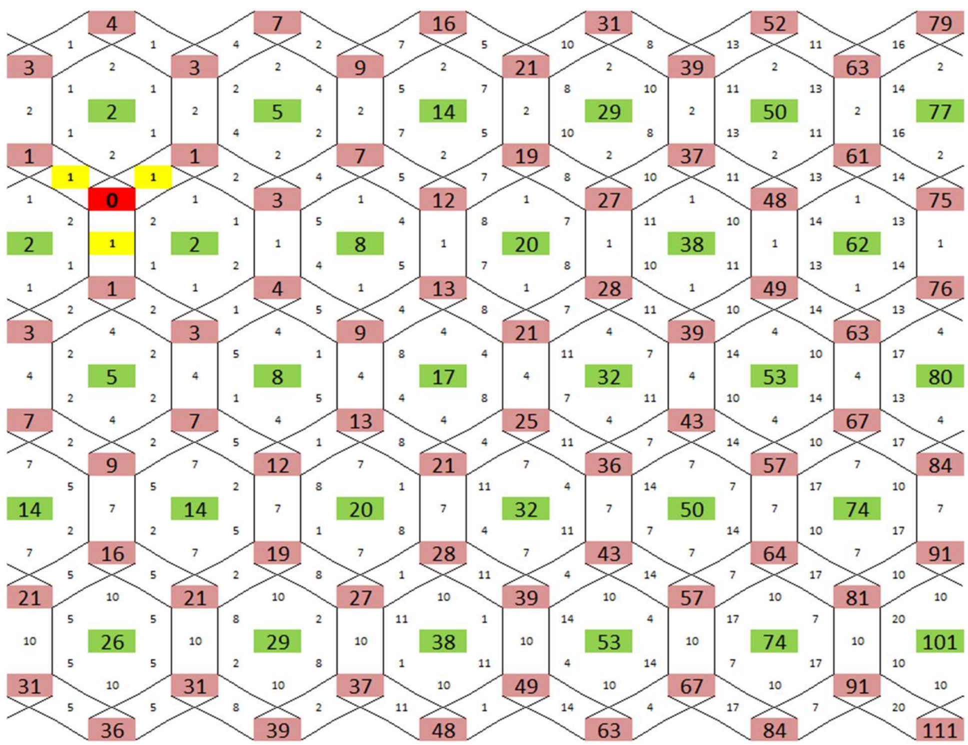

Figure 7: A "vertex first" projection.

1) Three slopes are all the same and not zero. This results in a single vertex touching the plane and the axis of symmetry of the parabolahedron being perpendicular to the plane at that point as in Figure 5.

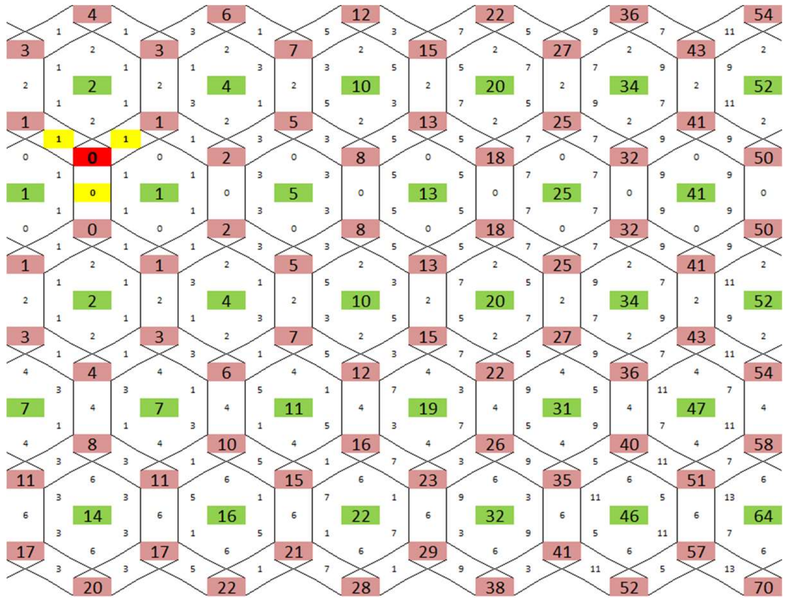

Figure 8: An "edge first" projection.

2) Two slopes are the same and the other is zero. This results in two vertices and their common edge (the one with zero slope) touching the plane and the axis of symmetry of the parabolahedron being perpendicular to the plane through the center of that edge as in Figure 6.

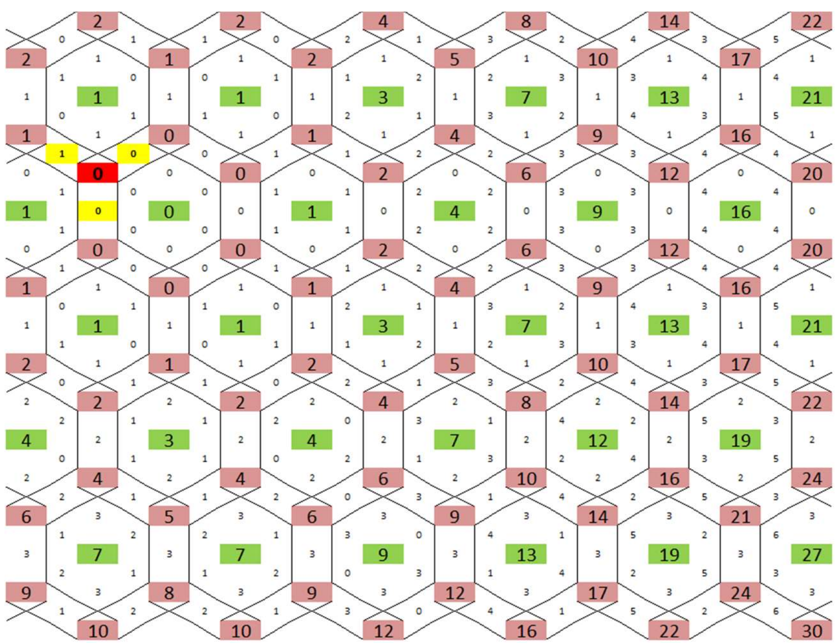

Figure 9: A "face first" projection.

3) Two slopes are zero and the other one is non-zero. This results in an entire hexagon touching the plane and the axis of symmetry of the parabolahedron being perpendicular to the plane through the center of that hexagon as in Figure 7.

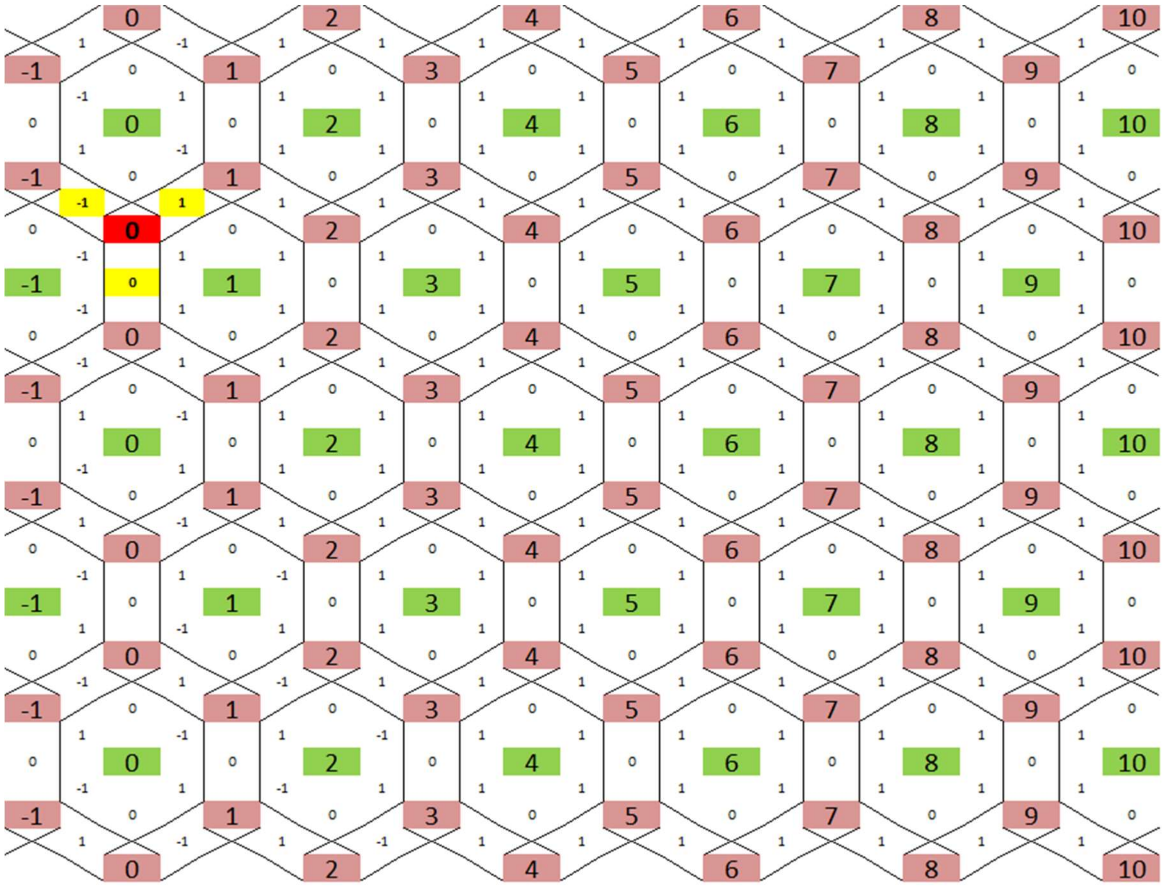

Figure 10: A plane is a degenerate parabolahedron.

4) One slopes is zero and the other two have the same absolute value, but with opposite signs. This results in a degenerate parabolahedron, in fact a plane, intersecting the original plane along the line extending from the origin and including the edge with 0 slope as in Figure 8.
5) Other variants exist including a variety of asymmetrical vertex first projections where the axis of symmetry is not through any of the vertices.
6) Using all negative slopes will generate an inverted parabolahedron with the bowl facing downward.
7) All zero slopes with a non-zero origin will result in a plane parallel to the original. This plane is another variant of a degenerate parabolahedron.

<u>Further study:</u>

The models generated by this algorithm can be used to feature many different aspects of mathematics and geometry.

In the simplest case, a predetermined grid like any of the examples shown can be the basis for stacking cereal or coins without even any focus on the resulting shape. Next in complexity might be the development of several different grid patterns like those featured here. Larger grids could be reproduced on posters allowing larger groups to collaborate on completing the cells. Further insights can be developed regarding rotational symmetries of the various patterns including 3-fold for the vertex first model, 2-fold or mirror symmetry for the edge first model, 6-fold symmetry for the face first model, as well as asymmetrical models. Intermediate studies can involve adding volumes of the hexagonal prisms to calculate a total volume of the parabolahedron. This is simpler than it may appear at first glance since the number in green represents the average height of all points on the facet, and therefore the volume of a given cell is simply the area of the hexagon multiplied by the number in green.

The model can also provide a centerpiece for more advanced level students to delve into discussions involving contour maps, calculating the slope of individual facets, especially those which don't lie on lines of symmetry. Other areas to be explored include exponential growth of the Z coordinates and vector addition of slope vectors in any given cell. Parallel slices of the parabolahedron can be discovered by observing concentric "rings" of hexagons around any chosen vertex or face, not just the origin.

Affine transformations and change of basis can be modeled by examining the cases where the three pairs of hexagon sides are different lengths, or the Z axis is not perpendicular to the grid. It may come as a surprise that the three cells surrounding the initial point are not necessarily coplanar as all our examples have been. These first three edges could be replaced by three non-coplanar axes such as the standard X, Y and Z axes which would result in an axis of symmetry which is not orthogonal to any of them. This model can even be used to generate a four-dimensional parabolahedron. Just drop me an email and ask me about it.

Enjoy and let me know what other insights you glean from this delightful geometric construct.

Honeycomb grid worksheet