Edgy Puzzles

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Countless puzzles involve decomposing areas or volumes of two or three-dimensional figures into smaller figures. “Polyform” puzzles include such well-known examples as pentominoes, tangrams, and soma cubes. This paper will examine puzzles in which the edge sets, or "skeletons," of various symmetric figures like polyhedra are decomposed into multiple copies of smaller graphs, and note their relationship to representations by props or body parts in dance performance.

The edges of the tetrahedron in Figure 1 are composed of a folded 9-gon, while the cube and octahedron are each composed of six folded paths of length 2. These constructions have been used in dances created by the author and his collaborators. The photo from the author’s 1997 dance “Pipe Dreams” shows an octahedron in which each dancer wields three lengths of PVC pipe held together by cord at the two internal vertices labeled *a* in the diagram on the right. The shapes created by the dancers, which might include whimsical designs reminiscent of animals or other objects as well as mathematical forms, seem to appear and dissolve in fluid patterns, usually in time to a musical score.

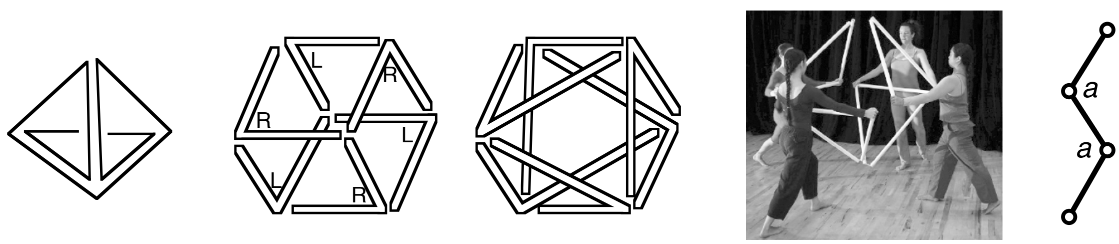
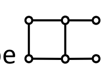
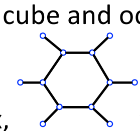


Figure 1. PVC pipe polyhedral skeletons used in dances

The desire in the dance company co-directed by the author to incorporate polyhedra into dance works led to these constructions, and to similar designs with loops of rope, fingers and hands, and the bodies of dancers. Just as mathematical concepts often suggest artistic explorations for those involved in the interplay between these fields, performance problems may suggest mathematical questions, in this case involving finding efficient and symmetrical ways to construct the skeletons, or edge sets, of the Platonic solids.

In one performance, we present an audience member with the puzzle of folding this shape , constructed from PVC pipe sections which fold at the vertices, into a tetrahedron. In the 2009 music and dance concert *Harmonious Equations* [2] we gave ourselves the puzzle of folding one shape wielded by three dancers into a cube and octahedron, and came up with a PVC pipe hexagon with pendant edges at each hexagon vertex,  which also folds to form a doubled-edge tetrahedron. In [4] the authors showed classroom activities involving making polyhedra with PVC pipe, fingers, and loops of string; George Csicsery documented the latter two of these in a series of short films [1]. In various papers the author investigated modular constructions of the Platonic solids in a manner reminiscent of modular origami: in [5] the author showed how to construct the five Platonic solids with six loops of three colors, in [3] with length six PVC pipe modules, and with the bodies of six dancers, and in [6] constructions of the Archimedean solids and various plane tessellations with one six-edge tree.

In this paper, we will explore a variety of puzzles derived from constructions like those described above, in this case using multiple copies of small trees made from paper straws. Similar puzzles can also be created with simple paper diagrams. Here's a simple example of five graphs called trees, several of which fold at the vertices to give the skeleton or edge set of a regular tetrahedron (which ones?). Here $T_n(a,b,c)$, for example, indicates a tree with n edges and pendant edges of lengths a, b, and c. (Note: this notation may not uniquely specify a graph for larger examples than we are considering here.)

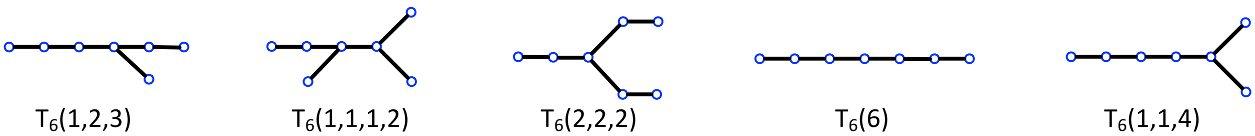


Figure 2. Trees which might fold to a tetrahedron (which ones?).

Over the last twenty years, since we began incorporating such polyhedra into our dance works, the author has created a variety of such puzzles, and I imagined it might be a good idea to find a way to market physical examples of the puzzles. Recently, however, I had an epiphany and decided to try to answer the question, "What would Mary Laycock do?" Mary Laycock was a pioneer In the use of manipulatives and physical activities in math classes. She wrote a book, *Straw Polyhedra*, which is still in publication, in which she showed how to use straws and bobby pins to construct polyhedra very simply and inexpensively. So, I decided to find a way to construct physical edgy puzzles for very little money, as a kind of homage to Mary Laycock.

By the way, Mary Laycock was a follower of Zoltan Dienes, a math educator who created numerous whole body and dance class activities for elementary and middle school students. Dienes was the son of Valeria Dienes, a prominent Hungarian dancer and choreographer who invented a somewhat mathematical dance notation. She also brought her family to live in the commune established in Greece by Isadora Duncan's brother, at which dance was an integral part of the schooling of young Zoltan. So, there's a nice dance history connection here as well!

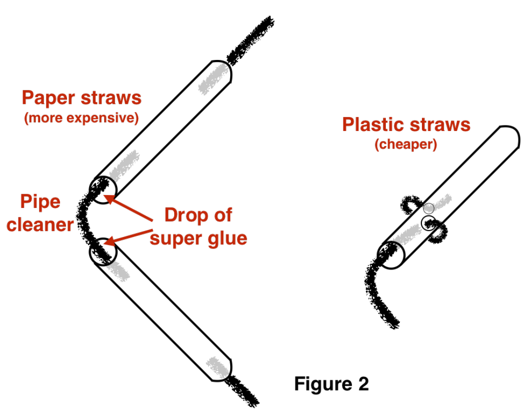


Figure 2

The simple construction method I've found most useful is to use paper straws for the edges, pipe cleaners to join them together at the vertices, and a drop of super glue to hold everything together (Figure 2). The pipe cleaners are flexible, yet hold their shapes, and the short pipe cleaner "tabs" at the ends allow the easy construction of three dimensional models.

I've found that paper straws are more expensive than plastic, but the glue does not hold to the plastic very well, and students playing with puzzles built with glue and plastic straws tend to pull them apart too easily! I have had some success punching holes in plastic straws and threading pipe cleaners through them, as shown on the right in Figure 2, but this is much more labor intensive than the method using paper straws. Glen Whitney (founder of MoMath) tells me that restaurants are beginning to replace plastic with paper straws, so we expect – or hope - that the price of paper straws will soon drop.

Puzzles
Below are a collection of puzzles that can be solved using the straw and pipe cleaner manipulatives or else using paper and pencil methods. Figure 3 shows how to use the paper and pencil puzzles to record a decomposition of the edges of the tetrahedron in Figure 3(b) using the tree in Figure 3(a). We will call that tree $T_6(1,2,3)$ since it has six edges and three sets of pendant edges of lengths 1, 2, and 3. Figure 3(b) is a puzzle diagram for the tetrahedron, and Figure 3(c) shows how we can draw over the diagram to solve the puzzle. Notice that we allow vertices to "overlap" or be identified, for example vertices A and B in the figure, but the edges must remain distinct.

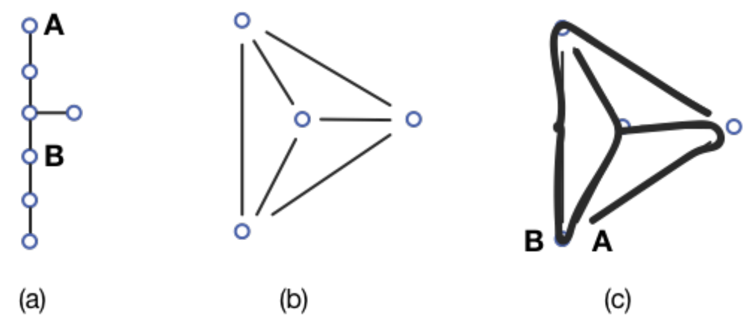


Figure 3

Alternatively, we might fold a regular tetrahedron out of a $T_6(1,2,3)$ tree made up of straws and pipe cleaners. A regular tetrahedron has edges that are all the same length, so the straw $T_6(1,2,3)$ will also have edges of equal length.

It turns out that $T_6(1,2,3)$ is a very versatile tree, as multiple copies of $T_6(1,2,3)$ will decompose the edges of each Platonic solid, each Archimedean solid, as well as each regular and semi-regular planar tessellation; see [6] for these decompositions. $T_6(1,2,3)$ will also decompose the edges of each of the Catalan solids, which are the duals of the Archimedean solids as well as the edges of many grid, cylinder, and toroidal graphs, some Johnson solids, and most duals of the semi-regular tessellations (contact the author for these solutions).

The grid graph P_m X P_n is the Cartesian product of the paths P_m and P_n with m and n vertices, respectively. The formal definition of the Cartesian product G X H of graphs G and H is the graph with vertices (u,v), where u and v are vertices in the graphs G and H, respectively; and with edges (u,v)(u'v'), where either u = u' and vv' is an edge in H, or v=v' and uu' is an edge in G.

Less formally, for the grid graph P_m X P_n we take a grid of m rows and n columns of vertices, with the vertices connected by edges in rectangular fashion (the graph in the upper left of Figure 4, for example, is P_3 X P_3). The graph C_n is the cycle with n vertices, and P_m X C_n is a "cylinder graph" or the skeleton of the n-prism. The graph C_m X C_n is known as a toroidal graph, since it is embeddable on the torus without edges crossing. In Figure 4 are a variety of somewhat easy decomposition puzzles; below each graph is the tree multiple copies of which will edge-decompose the graph. The bottom row shows the graphs P_2 X C_6 and P_2 X C_4. The edges of these graphs which extend out to the left we imagine connect to the rightmost pair of vertices. P_2 X C_4 is actually the cube.

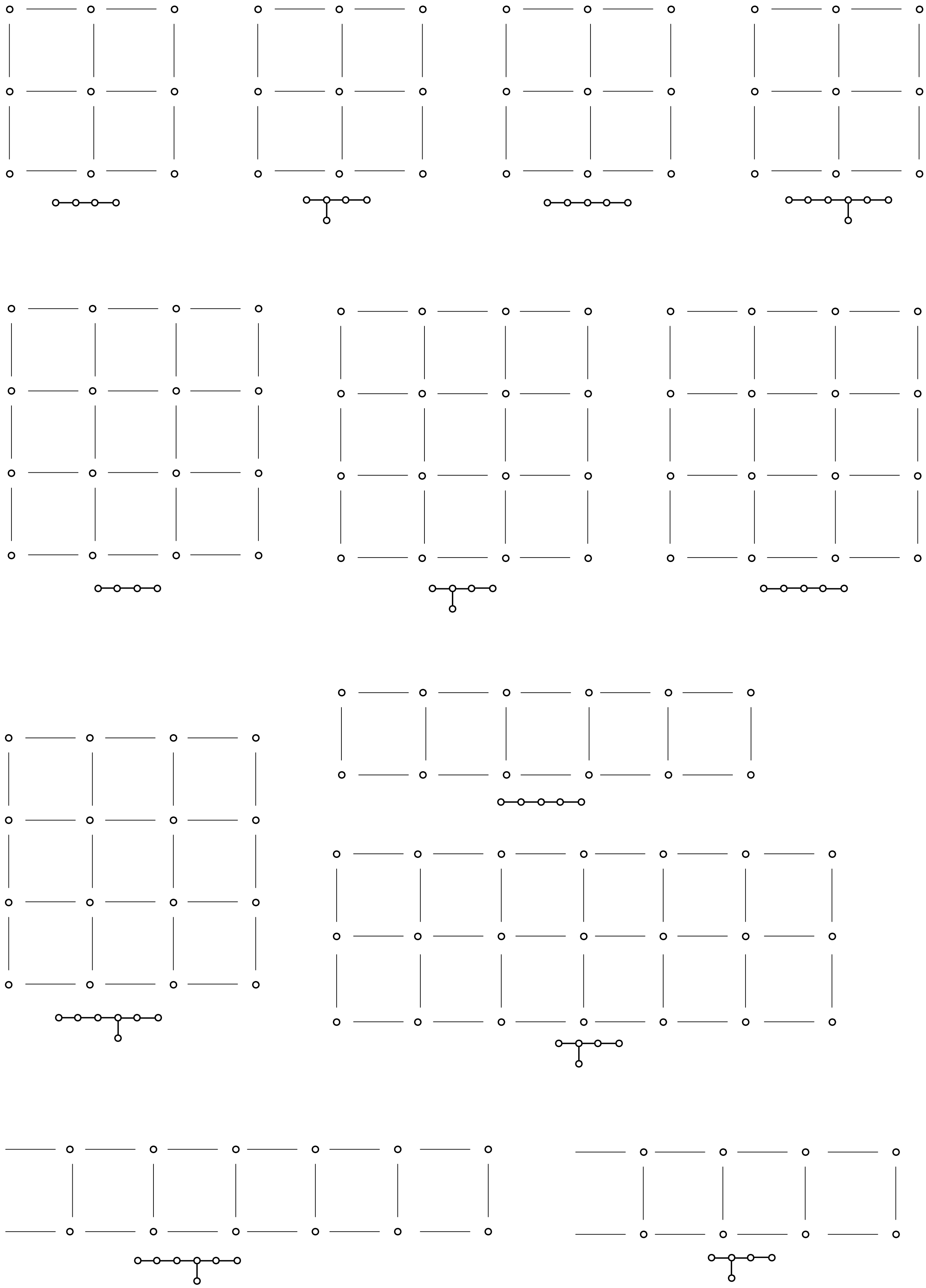


Figure 4

Many more such puzzles are easy to construct and solve. Here are a variety of extensions or generalizations of these decompositions which are all solvable and for which you might want to try to find solutions [7]. The "Examples" are small or initial cases which in some cases generalize easily, and many of which make enjoyable puzzles. Paper and pencil versions are included in Figures 4, 5, and 6.

Two-dimensional grids and cylinders.

P_n X P_{4k+n} by $T_4(1,1,2)$. Example P_3 X P_7, see Figure 6.
P_{4m} X P_{4n} by P_5. Example P_4 X P_4 on previous page.
P_{4m+2} X P_{4n+2} by P_5. Example P_6 X P_6, see Figure 6.
P_{3m} X P_{3n} by P_4. Example P_3 X P_3 on previous page.
P_{3m+1} X P_{3n+1} by P_4. Example P_4 X P_4 on previous page.
P_m X C_{4n} by P_5. Example P_3 X C_4, see Figure 6.
P_m X C_{4n} by $T_4(1,1,2)$. Example P_2 X C_4 on previous page.
P_2 X C_n by P_4. Example P_2 X C_4, the 3-cube, see Figure 6.
P_3 X C_{3n} by P_4. Example P_3 X C_3, see Figure 6.
P_3 X C_{4n} by $T_4(1,1,2)$. Example P_3 X C_4, see Figure 6.
P_4 X C_{3n} by P_4. Example P_4 X C_3, see Figure 6.
P_{3k+2} X C_n by P_4. Example P_5 X C_4, see Figure 6.
C_3 X C_{4n} by P_5. Example C_3 X C_4, see Figure 6.
C_n X C_{3n} by P_4. Example C_3 X C_4, see Figure 6.

Three-dimensional grids

P_2 X P_3 X P_{3k} by P_4, $k \geq 1$. Example P_2 X P_3 X P_3 by P_4, on next page.
P_3 X P_3 X P_n by P_4, $n \geq 1$. Example P_3 X P_3 X P_3 by P_4 on next page.
P_3 X P_3 X P_5 by P_5, see Figure 6.
P_4 X P_4 X P_{3k+1} by P_4, $k \geq 1$. Example P_4 X P_4 X P_4 by P_4.
P_2 X P_5 X P_5 by P_4, see Figure 6.

Mixed examples

P_2 X P_3 X P_3 by $6P_5 + 3P_4$, see Figure 6. (Endless similar possibilities - make some up!)
P_3 X P_4, with two colored edges decomposed by several trees, see top right of Figure 6.

In Figure 5 are a couple of three-dimensional paper and pencil grid puzzles plus a few polyhedra. For the polyhedra paper and pencil diagrams we do not insist that all edge lengths are equal, to facilitate drawing them efficiently; however, if building them out of straws and pipe cleaners the Platonic and Archimedean solids [6] are constructible with equal length straws. The "decastar," a decagon with length two paths attached to each decagon vertex, is what might be called an edge-GCD of the regular dodecahedron and icosahedron. An edge-GCD, or edge-greatest common decomposer, of two graphs G and H is a (not necessarily unique) graph with the largest number of edges that edge-decomposes both graphs. (In [6] the author showed that $T_6(1,2,3)$ is the unique edge-GCD of the five Platonic Solids.) The decastar may be easily modified to create a solution to a problem posed in Math Horizons to find a graph with the fewest number of leaves that decomposes both the dodecahedron and icosahedron [8]. In the diagram of the icosahedron on the next page, the vertices labeled *x* are " identified" and considered to be one vertex. Because the decastar easily decomposes into five copies of $T_6(1,2,3)$, the decastar's decomposition of the dodecahedron and icosahedron also provide decompositions by $T_6(1,2,3)$.

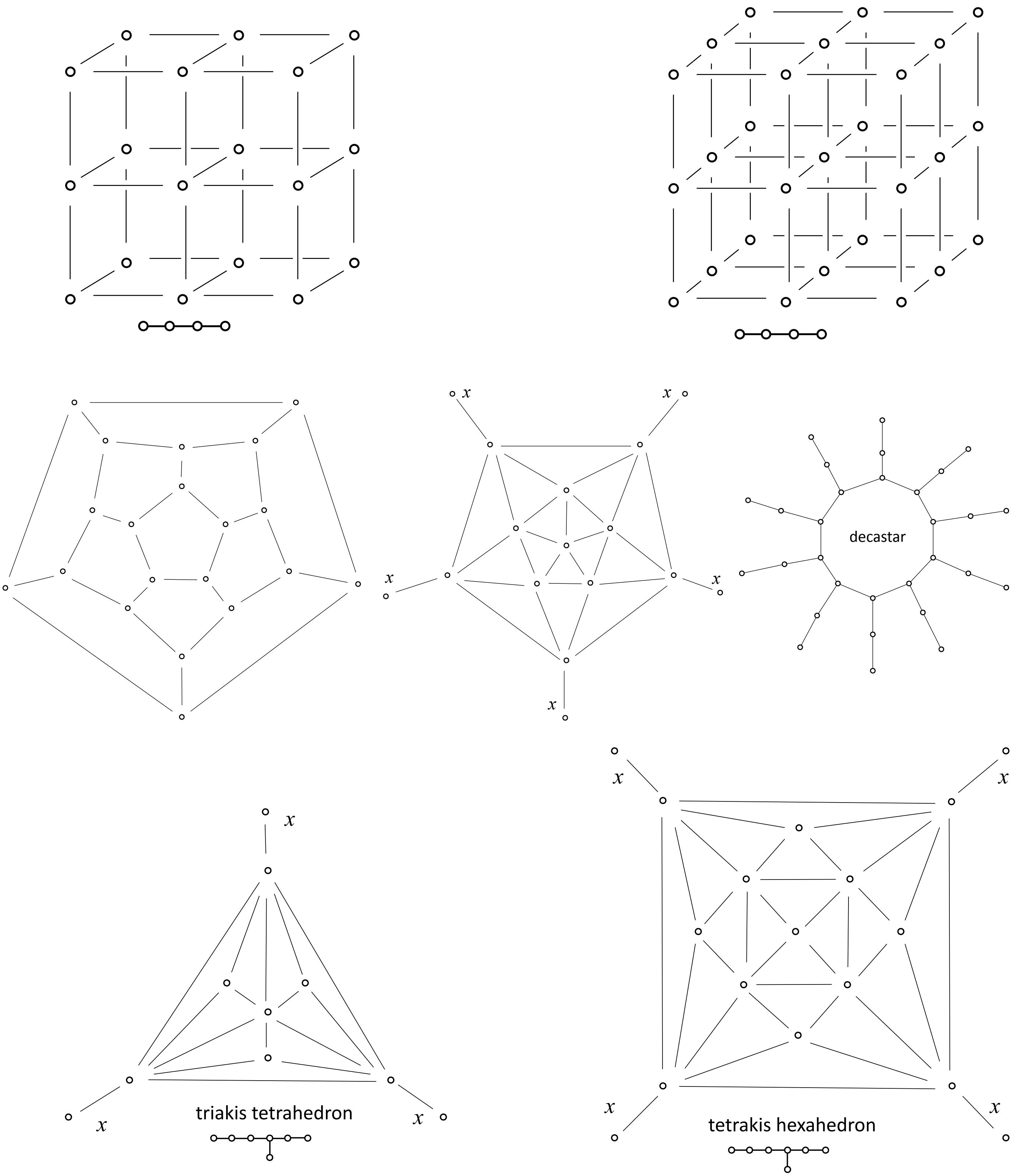


Figure 5

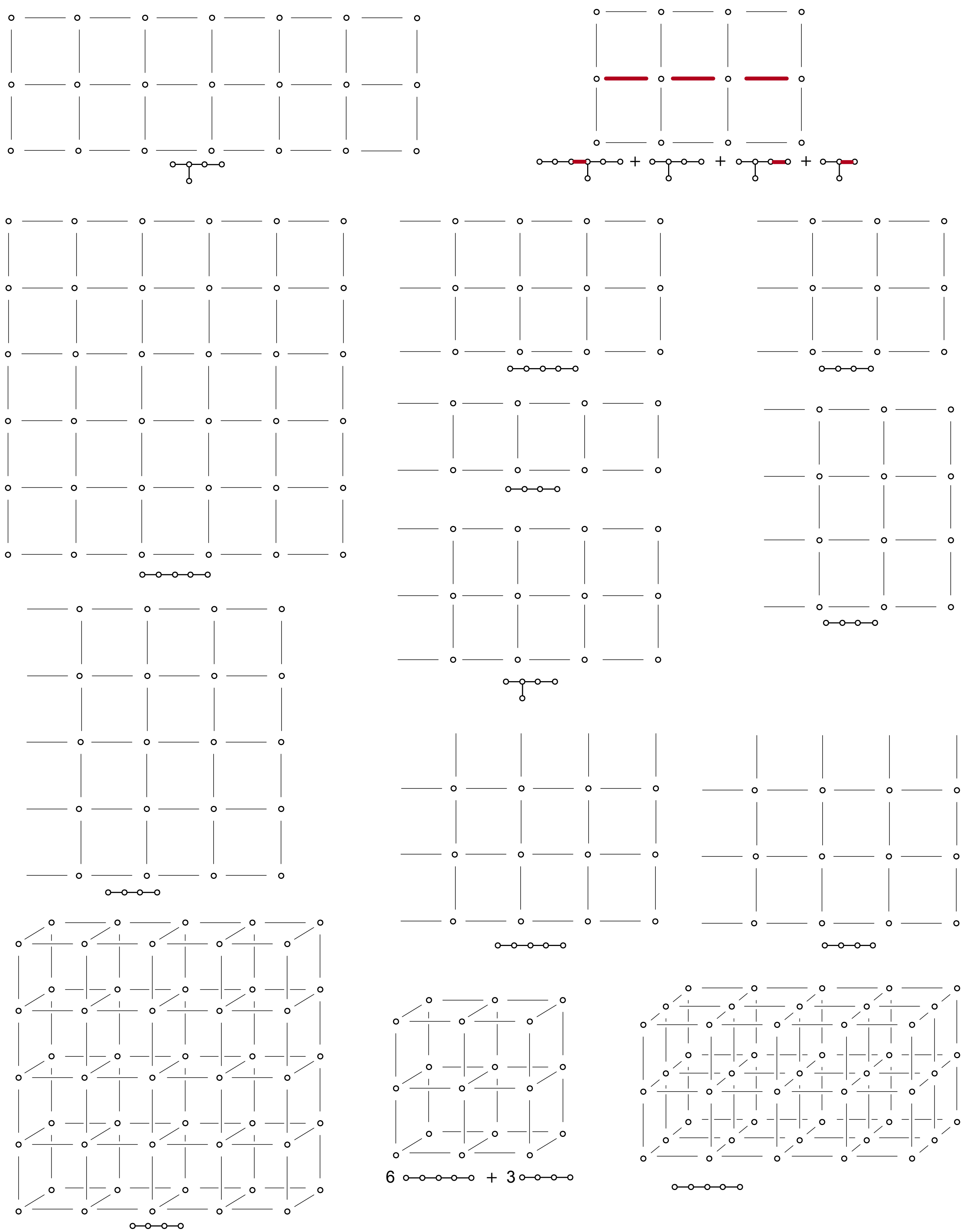


Figure 6

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