A Ternary Hamming Code in a Magic Trick

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For the Ternary Hamming Code trick, a magician is able to identify a number from 1 to 80 that a volunteer chose even if he decides to lie about one piece of information. It involves an error detection and correction algorithm, called *Hamming Code* (Hamming, 1950).

This ternary trick expands the binary version published on last Gathering for Gardner Exchange Book (Teixeira, 2018), where the volunteer could only pick a number from 1 to 15. Tricks with Hamming Code are common on the literature, for instance (Ehrenborg, 2006; Mateer, 2013; Teixeira, 2017), but this is the very first trick using a ternary version of Hamming Code. This trick was first presented during the 2018 MOVES conference, and the following description resembles (Teixeira & Park, 2020).

The Trick

Description: A volunteer thinks of a number from 1 to 80, he also selects a color from the rainbow (7 options). Then, the magician shows 7 colored cards with several numbers for the volunteer to say whether he sees his number or not, the volunteer lies on the card of his chosen color. The magician is able to find on which color the volunteer had lied, and then tell the chosen number.

Material: Copy and cut cards on appendix, if you have crayons you could color the cards accordingly. You could use actual fidget spinners of cut the ones on the appendix.

Preparation: Put the cards in order (red, orange, yellow, green, blue, magenta, purple). Practice how to check the options for ternary matching (see instructions below).

Performance: Gisele, the magician, will read Arthur's mind.

1. Gisele asks Arthur to pick a number between 1 and 80, and one of the colors of the rainbow (red, orange, yellow, green, blue, indigo or violet);
2. Gisele explains that Arthur has to say whether he can see his chosen number on each of the cards she shows. If the number appears, he has to say which color the number has on the card (black or red);
3. But Gisele also explains that Arthur should tell a lie on the card having the color he had chosen. The lie could be of any type: lying whether the number is on the card or not, or even lying about the color that the number has on that card;
4. For every time he says "red" for a card, Gisele lays the *fidget spinner* (see appendix) with its red circle pointing up. If Arthur says the number is "black" on a certain card, Gisele lays the *fidget spinner* with its black circle pointing up.

Finally, if he says that number is not on the card, she lays the *fidget spinner* with its white circle pointing up;

5. She arranges the *fidget spinners* side-by-side from left to right;
6. Once all seven cards are dealt, she looks at the *fidget spinners* and she can tell in which color the lie happened, which type of lie was told, and which number was selected.

Trick: Trick is based on the ternary extension of the Hamming Code. The first four cards resemble a ternary-digit trick with numbers 1 to 80. Having the color *red* means the correspondent position has digit 1, color *black* means that position has digit 2, while if the number does not appear (fidget spinner has white circle pointing up), then the correspondent position would be zero. If there were no lies allowed, then we'd only need the first four cards. Simply, we would add the top left number on each color (red or black) for the corresponding cards in which the volunteer claims to see the number.

Summary:

- Don't see the number (color is white) = 0.
- The number is red = 1.
- The number is black = 2.

However, a lie was told and we are also trying to discover where the lie happened, and which type of lie it was.

We use the following system: if no lie was told, then the result would be that the 7 positions would satisfy the following relationships.

- $\left(\text{position}_2 + \text{position}_3 + \text{position}_4\right) = \text{position}_5 \pmod 3$
- $\left(\text{position}_1 + \text{position}_3 + \text{position}_4\right) = \text{position}_6 \pmod 3$
- $\left(\text{position}_1 + \text{position}_2 + \text{position}_4\right) = \text{position}_7 \pmod 3$

The first four cards will serve to compute the chosen number by determining what is the ternary expansion of the chosen number.

The last three cards are the "checking digits".

During the trick, for each of the checking positions (positions 5, 6 and 7), the magician needs to mentally compute the *discrepancy*:

- $Discrepancy_5 = \left(\text{position}_2 + \text{position}_3 + \text{position}_4\right) - \text{position}_5 \pmod 3$
- $Discrepancy_6 = \left(\text{position}_1 + \text{position}_3 + \text{position}_4\right) - \text{position}_6 \pmod 3$
- $Discrepancy_7 = \left(\text{position}_1 + \text{position}_2 + \text{position}_4\right) - \text{position}_7 \pmod 3$

And while displaying the *fidget spinners* for each position, if a discrepancy is detected, then:

- If discrepancy is 1: let the *fidget spinners* be displayed a little higher than others (in such way that the magician can see, but it would not call the audience's attention);
- If discrepancy is 2: let the *fidget spinners* be displayed a little lower than others.

According to the number of discrepancies, the position of the lie can be determined by similar analysis as the binary case (**?**). Once the position of the lie is identified, the sum of the value of the discrepancies and the value represented by the lie (the color of the *fidget spinner*: white=0, red=1, black=2) will identify the type of lie.

To find the *position* of the lie:

There are discrepancies on:	Then lie was on:
5	5
6	6
7	7
5 and 6	3
5 and 7	2
6 and 7	1
5, 6 and 7	4

Once the position of the lie is known:

- If there was only one discrepancy: the corresponding card is the lie, and the actual value of the card is supposed to the value of the *discrepancy* added to the *lie-value* (the value that corresponds to the *fidget spinner*'s color) $\pmod 3$.
- If there were more than one discrepancy, then they all have the same value (1 or 2):
  - the value of each discrepancy and the lie-value adds up to three: then the person lied about the color; otherwise
    - if the lie-value is not zero: the true value is zero;
    - if the lie-value is zero: the true value is "3 minus discrepancy".

Explanation: It is based on the theory developed in the exercises.
Hint: Practice the error recognition. At first, it may take you a while to figure out the position of the lie and its type. Onde identified, fix the lie, before telling the chosen number.

Examples

Example 1: Suppose that the chosen number is 37, and the chosen color is magenta (the sixth card).

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Red	Yes, Red	No	Yes, Red	Yes, Black	Yes, Red (lie)	No
1	1	0	1	2	1	0

- $\textit{Discrepancy}_5 = \big(\text{position}_2 + \text{position}_3 + \text{position}_4\big) - \text{position}_5 \pmod 3 = 1 + 0 + 1 - 2 = 0$ (no discrepancy)
- $\textit{Discrepancy}_6 = \big(\text{position}_1 + \text{position}_3 + \text{position}_4\big) - \text{position}_6 \pmod 3 = 1 + 0 + 1 - 1 = 1$ (discrepancy)
- $\textit{Discrepancy}_7 = \big(\text{position}_1 + \text{position}_2 + \text{position}_4\big) - \text{position}_7 \pmod 3 = 1 + 1 + 1 - 0 = 3 = 0 \pmod 3$ (no discrepancy)

Since, there is only one discrepancy, then that's where the lie is. The color of that card was supposed to be $1+1$ (the actual value plus the value of the discrepancy), hence color 2 (black). The chosen number is $1 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 1 \times 3^0 = 27 + 9 + 0 + 1 = 37$.

Example 2: Suppose that the chosen number is 70, and the chosen color is orange.

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Black	No (lie)	Yes, Black	Yes, Red	Yes, Red	Yes, Black	Yes, Red
2	0	2	1	1	2	1

- $Discrepancy_5 = \big(\text{position}_2 + \text{position}_3 + \text{position}_4\big) - \text{position}_5 \pmod 3 = 0 + 2 + 1 - 1 = 2$ (discrepancy)
- $Discrepancy_6 = \big(\text{position}_1 + \text{position}_3 + \text{position}_4\big) - \text{position}_6 \pmod 3 = 2 + 2 + 1 - 2 = 3 = 0 \pmod 3$ (no discrepancy)
- $Discrepancy_7 = \big(\text{position}_1 + \text{position}_2 + \text{position}_4\big) - \text{position}_7 \pmod 3 = 2 + 0 + 1 - 1 = 2$ (discrepancy)

Since, checking digits 1 and 3 show discrepancy, the lie is on the second card (orange). Since he told 0, the correct value was ”3 minus discrepancy”: $3 - 2 = 1$ (red). The chosen number is $2 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 54 + 9 + 6 + 1 = 70$.

Example 3: Suppose that the chosen number is 16, and the chosen color is red (the first card).

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes, Black (lie)	Yes, Red	Yes, Black	Yes, Red	Yes, Red	No	Yes, Black
2	1	2	1	1	0	2

- $Discrepancy_5 = \big(\text{position}_2 + \text{position}_3 + \text{position}_4\big) - \text{position}_5 \pmod 3 = 1 + 2 + 1 - 1 = 3 = 0 \pmod 3$ (no discrepancy)
- $Discrepancy_6 = \big(\text{position}_1 + \text{position}_3 + \text{position}_4\big) - \text{position}_6 \pmod 3 = 2 + 2 + 1 - 0 = 5 = 2 \pmod 3$ (discrepancy)
- $Discrepancy_7 = \big(\text{position}_1 + \text{position}_2 + \text{position}_4\big) - \text{position}_7 \pmod 3 = 2 + 1 + 1 - 2 = 2$ (discrepancy)

Since, checking digits 2 and 3 show discrepancy, the lie is on the first card (red). Since the value of the discrepancy is 2 and he told 2, the correct value was ”3 minus discrepancy”: $2 - 2 = 0$ (white). The chosen number is $0 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 0 + 9 + 6 + 1 = 16$.

References

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Appendix

Ternary Hamming Code Cards

27	28	29	30	31	32
33	34	35	36	37	38
39	40	41	42	43	44
45	46	47	48	49	50
51	52	53	54	55	56
57	58	59	60	61	62
63	64	65	66	67	68
69	70	71	72	73	74
75	76	77	78	79	80

9	10	11	12	13	14
15	16	17	18	19	20
21	22	23	24	25	26
36	37	38	39	40	41
42	43	44	45	46	47
48	49	50	51	52	53
63	64	65	66	67	68
69	70	71	72	73	74
75	76	77	78	79	80

3	4	5	6	7	8
12	13	14	15	16	17
21	22	23	24	25	26
30	31	32	33	34	35
39	40	41	42	43	44
48	49	50	51	52	53
57	58	59	60	61	62
66	67	68	69	70	71
75	76	77	78	79	80

1	2	4	5	7	8
10	11	13	14	16	17
19	20	22	23	25	26
28	29	31	32	34	35
37	38	40	41	43	44
46	47	49	50	52	53
55	56	58	59	61	62
64	65	67	68	70	71
73	74	76	77	79	80

1	2	3	4	6	8
9	10	12	14	16	17
18	20	22	23	24	25
28	29	30	31	33	35
36	37	39	41	43	44
45	47	49	50	51	52
55	56	57	58	60	62
63	64	66	68	70	71
72	74	76	77	78	79

1	2	3	4	6	8
10	11	12	13	15	17
19	20	21	22	24	26
27	28	30	32	34	35
36	37	39	41	43	44
45	46	48	50	52	53
54	56	58	59	60	61
63	65	67	68	69	70
72	74	76	77	78	79

1	2	4	5	7	8
9	10	12	13	15	16
18	20	21	23	24	26
27	28	30	31	33	34
36	38	39	41	42	44
46	47	49	50	52	53
54	56	57	59	60	62
64	65	67	68	70	71
72	73	75	76	78	79

Ternary Hamming *Fidget Spinners*