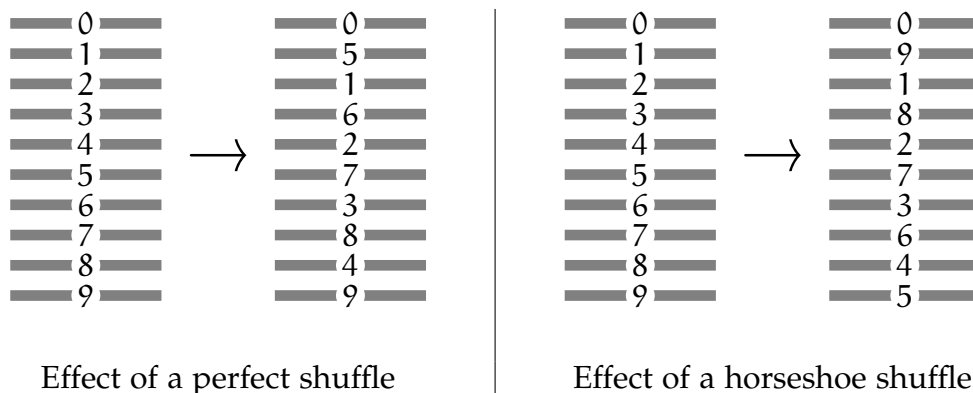


# A card trick inspired by perfect shuffling

Steve Butler\*

## Perfect shuffles and horseshoe shuffles

In perfect shuffles we take a deck, split it exactly in half, and then interleave cards from the two sides. This can be done in two different ways and can be distinguished by what happens to the top card (termed **in** and **out**). A variant of this has recently been studied called the horseshoe shuffle which adds one ingredient, namely that before interleaving we reverse the order of one of the halves. The name horseshoe is connected to Smale's horseshoe map, for more information on the mathematics of this shuffle see [1].



From a mathematical perspective there is a simple connection between the horseshoe shuffle and the perfect shuffle of a deck with twice as many cards. In other words the theory works more or less the same as before. From a performance perspective though there is a strong advantage to working with the horseshoe shuffle. Namely, these involve shuffles that are already named and easier to perform: *milk shuffle* and *Monge shuffle*. So makes for easier tricks to teach people and have them perform. We will give one such example here.

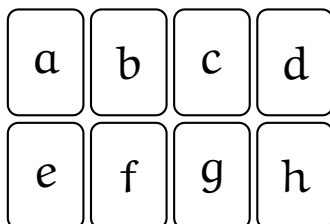
## The effect

Take the cards **A, 2, ..., 8** and lay them out on the table so that the audience member can see them and announce "We are going to practice a few basic card shuffling techniques used by magicians."

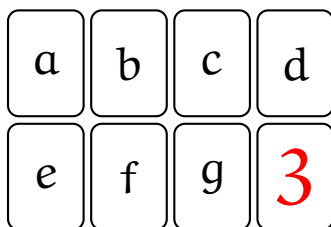
\*Iowa State University, Ames, IA 50010. butler@iastate.edu

The cards are now picked up<sup>1</sup> by the performer and they continue, “There are many different shuffles that are used in magic the first one is very simple, we call it dealing down.” The performer now deals down one card at a time and picks up the stack. “Of course we can deal down more than one card at a time, since there are eight cards in this stack we can deal down anything which divides eight so we deal down two at a time.” The performer deals down two at a time and picks up the stack. “Of we can deal down four at a time.” The performer deals down four at a time and picks up the stack. “We can even deal down eight at a time...” The performer places the stack on the table and picks it up again. “...but we usually don’t use that one. “Next we have the milk shuffle.” The performer demonstrates the milk shuffle (pulling off the top and bottom cards together and placing it on the table and repeating) and picks up the stack. “Finally we have the Monge shuffle.” The performer demonstrates the Monge shuffle (moving the cards from one hand to the other, one card at a time alternating above and below). “And the Monge has two variations.” The performer demonstrates the other variant (i.e., switch the order of what goes above and below).

At this time the performer hands the deck to the audience member and asks them to now practice the shuffles in any order they want. The performer can have a stunt deck handy to help them remember how to do the various shuffles. Once the audience member is convinced it is well shuffled they are asked to deal the cards face down in the following pattern: left to right, top to bottom (i.e., a, b, . . . , h as shown below).



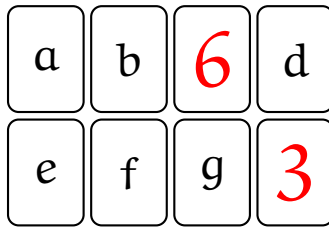
“I am going to try and figure out your cards.” The performer now appears to exert some mental energy, but fails. “This works a lot better when I use a marked deck. I need some help, maybe a hint, pick any card you want and turn it over.” Suppose the audience member now turns a card over and the performer now sees the following.



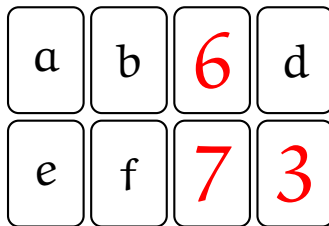
“Ah, the 3, this is helping, the fogs are starting to lift. Since we are doing mathematics let’s look for a number that three divides into.” The performer points to the third card in the top row and declares, “This card is the 6.” The audience member turns it over and sees it is correct.

---

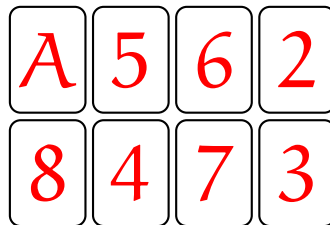
<sup>1</sup>With a small sleight of hand move to be discussed later



The performer continues, "I am almost there, just one more hint and I should be able to discern the rest. How about we turn over one of these two cards." The performer points at the cards in positions d and g and suppose the audience member turns over the card in position g to reveal a 7.



"I see it now! The fog has lifted." The performer now starts pointing at cards and declares what they are, the audience member turns them over discovering that the performer guesses them all correctly!



## How it works

There are several important ingredients to this trick. First and foremost is the fact that this is being done with *eight* cards. What makes this important is that eight is a power of two and it is well known that for powers of two the number of arrangements that happen under perfect shuffles is dramatically smaller than would we expected. For example for eight cards there are  $8! = 40320$  different arrangements; but using the shuffles outlined above there will be only 32 different possibilities (a much easier number to handle!). These possibilities follow very specific rules in their structures (see [1]); every shuffle outlined above (and a few more) will preserve this structure. Indeed this is in essence the whole reason why this trick works.

The other important ingredient for us will be the use of binary numbers. We will represent each number as a three-digit binary number (with leading 0's if needed), and we will have 8 correspond to the number 0. So in particular we have the following pairings:

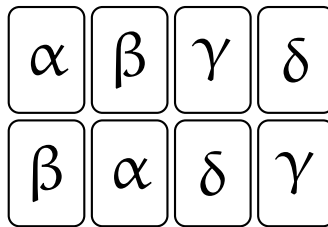
$$\begin{array}{llll}
 8 \leftrightarrow 000_{(2)} & A \leftrightarrow 001_{(2)} & 2 \leftrightarrow 010_{(2)} & 3 \leftrightarrow 011_{(2)} \\
 4 \leftrightarrow 100_{(2)} & 5 \leftrightarrow 101_{(2)} & 6 \leftrightarrow 110_{(2)} & 7 \leftrightarrow 111_{(2)}
 \end{array}$$

## The setup

The first part of the trick is to get the cards in the right order. From the above we see that if we left the cards in the order  $A, 2, \dots, 8$  that we would have them in the wrong order in terms of binary. So the one “sleight of hand” is to move the  $8$  card to the other end of the deck. One easy way to do this is to spread the cards out in order then as you grab the cards you “mistakenly” only grab the first seven cards and then pick up the last card and put it back in the deck (now in the right place).

## The first reveal

For all of the 32 possible orderings that can happen the cards are naturally pairing in two ways. One is location, and the other is value. So if we know one card, then we know the location and value of its pair. For location we have the following (essentially notice that this forms a pair of X's).



For the values we pair based off of the binary numbers by the following rule: flip the first and last bit. Let's denote this rule as  $*-*$ , i.e., a  $*$  indicates that we flip the corresponding bit and a  $-$  indicates that we keep the bit the same. So our four pairs are as follows (of course the performer should embellish as to why these go together):

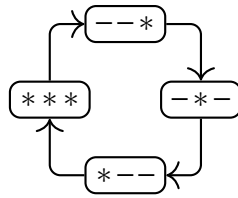
$$\begin{array}{ll} \{A = 001_{(2)}, 4 = 100_{(2)}\} & \{2 = 010_{(2)}, 7 = 111_{(2)}\} \\ \{3 = 011_{(2)}, 6 = 110_{(2)}\} & \{5 = 101_{(2)}, 8 = 000_{(2)}\} \end{array}$$

Combining location and value we readily can determine where the other pair is at for whichever card the audience member turns over.

## The second reveal

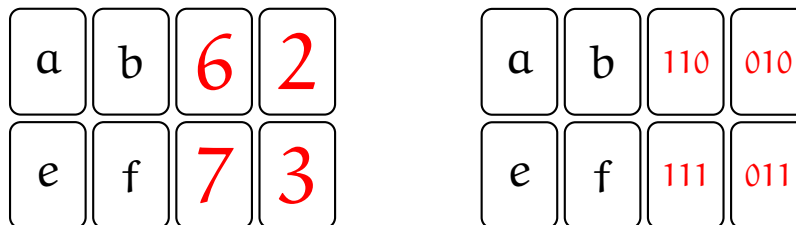
The performer gets the audience member to turn over another card, one which is from a pair that would form a square with the current pair. Using the same technique as from the first reveal we would now have one half of the cards.

To wrap it up we now dip our toes a little more into the structure of how the cards are related. In particular if we look at either of the now uncovered horizontal pairs we would have that there are *four* possible ways that the numbers relate in binary as shown by the boxes in the following diagram.

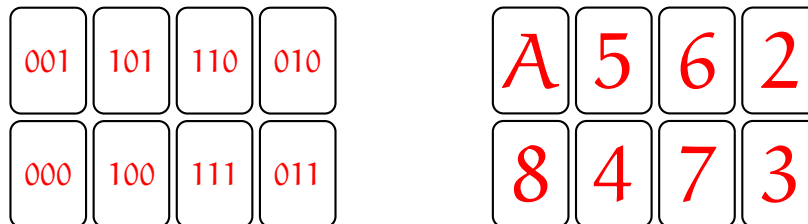


Whichever way the cards are related in binary (horizontally) follow the arrow to the next box and that will show how the revealed square of shown cards connects to the still hidden set of cards.

This is perhaps best seen by example. So if we go to back to the point in the example performance where the second card has been revealed and apply the first reveal rule we would see in binary that we have the following.



At this point we see that (horizontally) the revealed cards are related by switching the leading bit, i.e.,  $*--$ . So following the arrow in the diagram we see that the left and right sides are connected by flipping all bits, i.e.,  $***$ . Carrying this out we get the following in binary which we can easily convert back.



## Notes

This is quite simple and can be learned and taught quickly.

This does take some mild practice to do the reveals mentally. For beginning it is useful to have the following written down: (1) the binary relationships between cards and numbers (8 being 0 is important); (2) the  $*-*$  pattern of the first reveal; (3) the four patterns and their connections for the second reveal. All this becomes natural with just a bit of practice.

For more information about the mathematics behind this trick look at [1].

## References

- [1] Steve Butler, Persi Diaconis, and Ron Graham, The mathematics of the flip and horseshoe shuffles, *American Mathematical Monthly* **123** (2016), 542–556.