Sketching a Projectile on a Ramp

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Abstract

This informal paper presents a circle-and-line method for constructing the trajectory of a projectile bouncing up and down a ramp. The method is presented as a series of collaborative discoveries.

*Latent structure is master of obvious structure.* —Heraclitus

1 A Second Reflection

Raise a cannon halfway to vertical and fire. The cannonball flies over to a small trampoline, bounces, and retraces its path back to the cannon.

 1) What is the angle of the trampoline?
 2) What other angle will work?

Solution: A trampoline at 45° will return the ball to the cannon, clearly, but half that angle will also work: the ball will fly over to the trampoline, bounce vertically, then retrace its path back to the cannon.
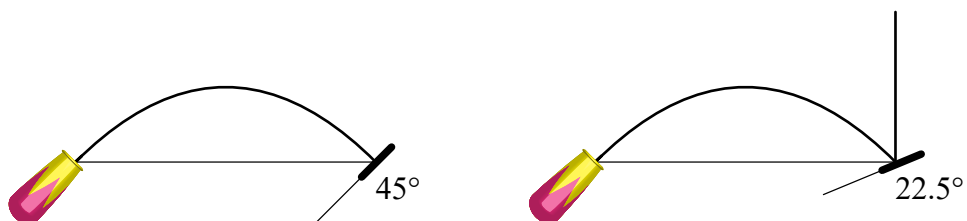
Nick McKeown (Stanford CS) shared this problem with me back in 2010, and I immediately featured it in The Times Numberplay column. Readers readily identified these two angles. But why stop there?

2   Seeking Structure

It was clear that 45° and 22.5° were not the only solutions—with a long enough trampoline, infinitely many angles would work. But what was the underlying pattern?

In an ongoing Numberplay discussion, the team of Dr. W, Marco Moriconi, Tudor, Hans Chen, Pummy Kalsi, and Pradeep Mutalik settled on the fire-from-ramp approach and computed a number of angles and distances between bounces. Nick Baxter created the corresponding images of the flight paths. Peter Norvig, director of research at Google, created a useful projectile/ramp simulation tool.

http://norvig.com/inclined_plane.html
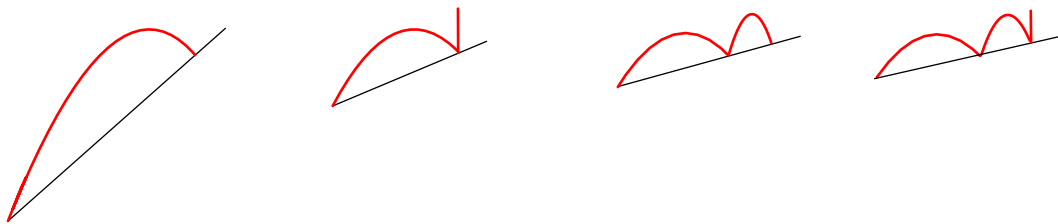
The Method of Apollonius

A parabola's envelope of tangent lines can be created by taking a sloping line segment, mirroring it, dividing each by any number of equal segments, then connecting as follows.
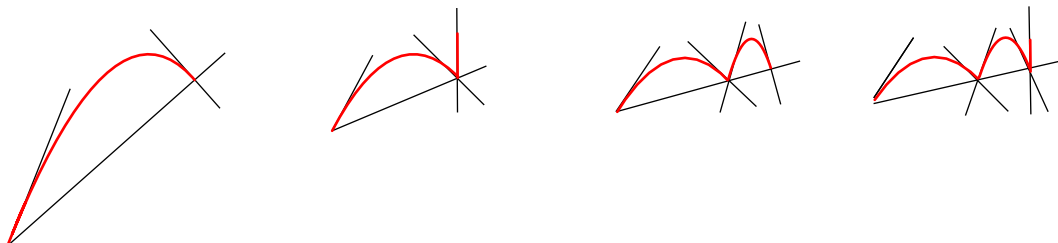
The Nine-Dot Problem

Nine dots arranged in a grid can be connected with a series of connected straight line segments as follows—the iconic outside-box solution.
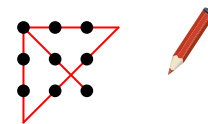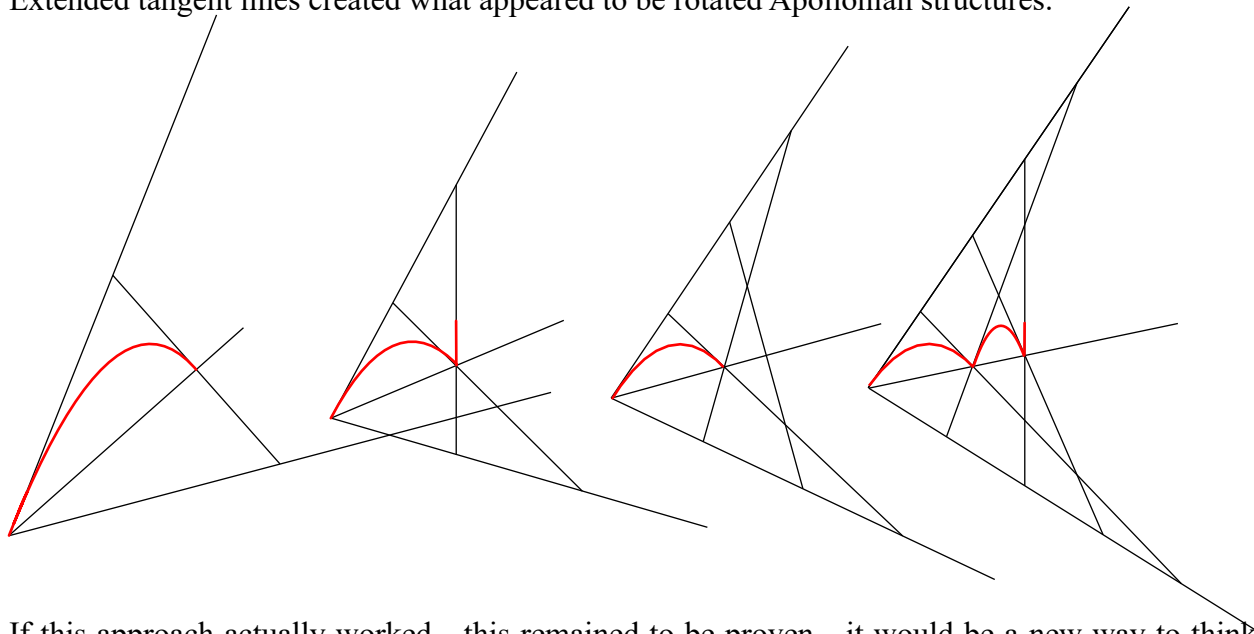
It would seem that the underlying structure would have to include the following tangent lines. What form would these lines take if extended?

3 Apollonius Sideways

Extended tangent lines created what appeared to be rotated Apollonian structures.

If this approach actually worked—this remained to be proven—it would be a new way to think about the path of a projectile on an inclined plane. What the logic behind these structures?

Key to the structures were the sideways V shapes, which could be divided into 2, 3, 4, 5 segments to create tangents for 2, 3, 4, 5 bounces. How could these be derived?
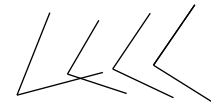

For ideas I turned to Nick Baxter, who suggested throwing the problem to geometers. George Hart came to mind. We phrased the challenge this way:

> **What Angles?**
> Place a cannon directly on a ramp. What launch and ramp angle will cause a ball to bounce up the ramp n times before reversing direction?
>
>
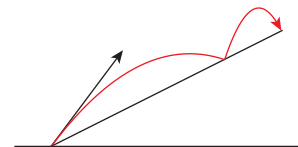
George's conclusion: For a ramp angle θ and incremental launch angle α, the number of bounces up a ramp is 1/2tanθtanα. (Full solution in appendix). This was exactly what I was looking for.
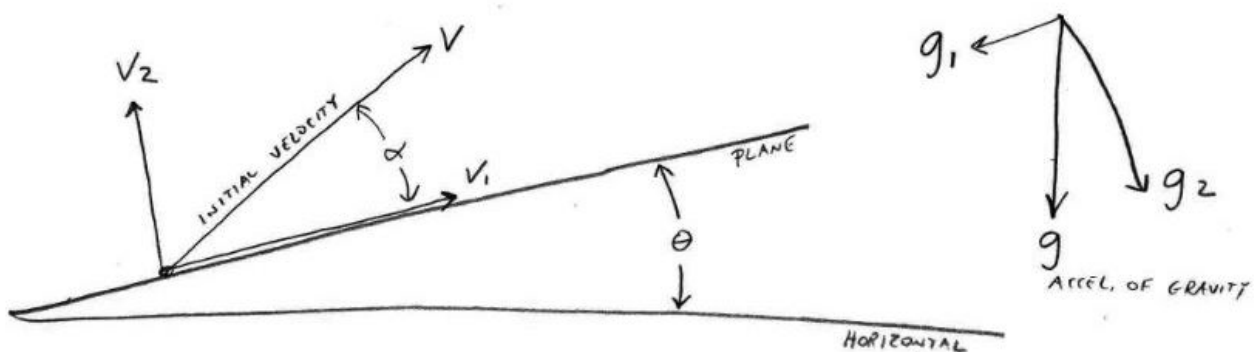
3   A Deeper Structure

The structure was now clear. The V shapes had openings of 2θ and were tilted at angle α from horizontal, where $\tan\theta\tan\alpha = 1/\text{bounces}$. (There are several ways to add bounce detail.)

Below are three possible θ and α combinations for 1, 2, and 3 bounces, with the 3-bounce scenarios fully worked out (there are several ways to construct bounce-level detail). I find circle constructions practical, precise, and aesthetically pleasing.
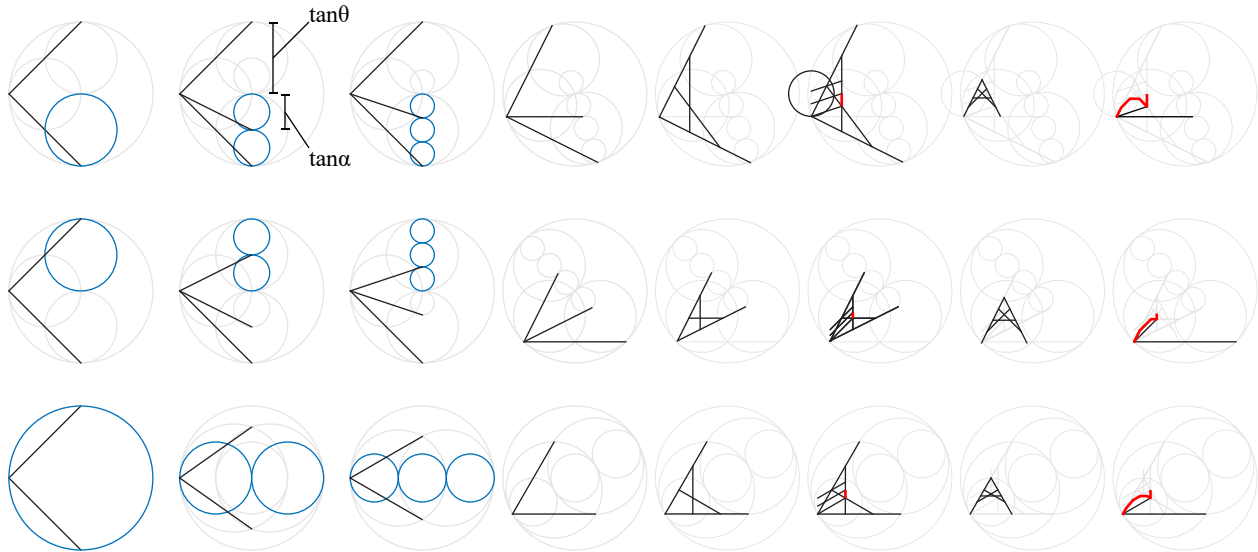
This was the deeper structure I had been seeking. The basic idea could be used to easily and accurately construct not only the launch/ramp angles to generate any number of bounces, but also estimate the number of bounces for any launch/ramp configuration, accurately determine the path of a ball bouncing down a ramp, and determine the trajectory of a ball fired in any direction based on a single vertical launch. The reflection was complete.
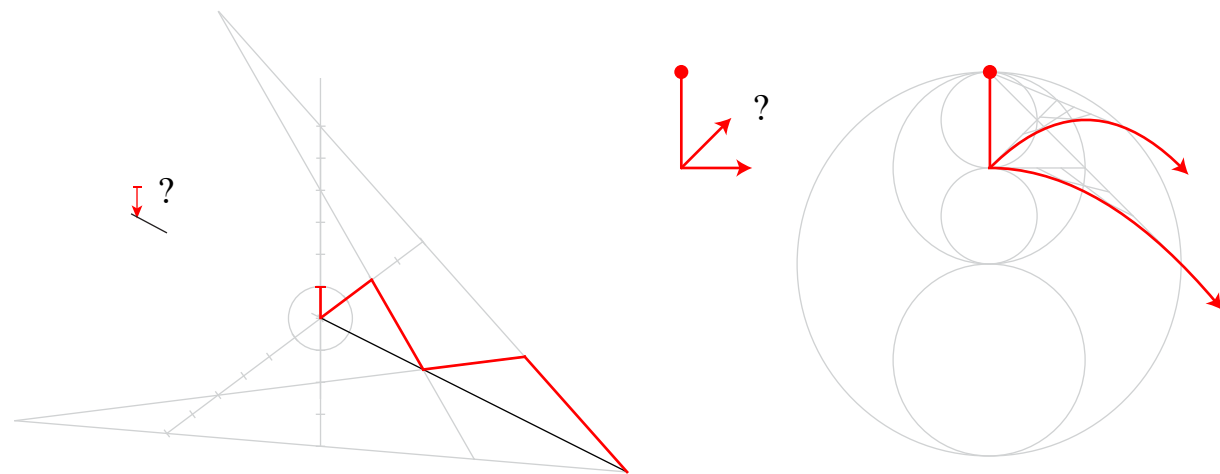
Appendix 1: George Hart's Solution
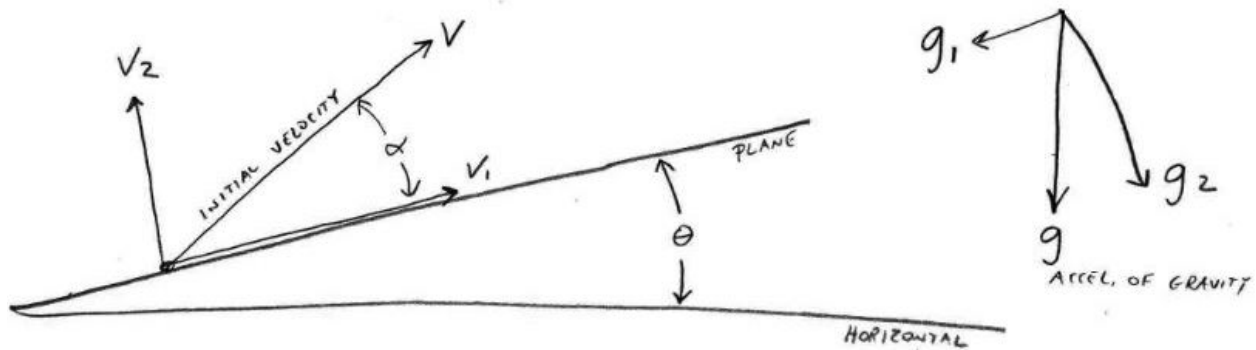
$\begin{bmatrix} \text{KNOWING} \quad \Delta V = a\tau, \quad \text{and separating into components 1 \& 2 in} \\ \text{direction of plane surface and normal to it.} \end{bmatrix}$

$V = $ initial velocity with components $\quad V_1 = V\cos\theta, \qquad V_2 = V\sin\theta$

$g = $ accel. of gravity with components $\quad g_1 = g\sin\alpha, \qquad g_2 = g\cos\alpha$

τ_R (RISE TIME) SATISFIES $\quad g_2 t_R = V_2 \qquad$ so $\quad \tau_R = V_2/g_2$

τ_B (BOUNCE TIME) $= 2\tau_R = 2V_2/g_2$

τ_S (time until V_1 motion STOPS) satisfies $\quad g_1 \tau_S = V_1 \qquad$ so $\quad \tau_S = V_1/g_1$

n (number of bounces while contact point move rightward)

$$n = \tau_S/\tau_B = \frac{V_1/g_1}{2V_2/g_2} = \frac{V\cos\theta \; g\cos\alpha}{2V\sin\theta \; g\sin\alpha} = \frac{1}{2\tan\theta\,\tan\alpha}$$

$\begin{bmatrix} \text{THIS ASSUMES ALL THE USUAL THINGS ABOUT ZERO AIR FRICTION, PERFECTLY} \\ \text{ELASTIC COLLISIONS, PARALLEL CONSTANT GRAVITY, ETC.} \end{bmatrix}$

G. HART
Nov 2012

Appendix 2: Proof

We seek to prove that the intersection points between the construction shown in black and a line bisecting the angle at the bottom left vertex (in blue) are spaced quadratically with the zero point at the single perpendicular intersection.
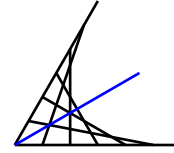
Let N be number of subdivisions of the bounding lines such that the subdivision points are equidistant and N is one more than the number of lines between these subdivision vertices. In the above case, there are 5 lines and N = 6. Labelling the subdivision vertices by their distance from the vertex (down to arbitrary scale factor) such that the bottom left vertex is 0 and the end of each bounding line is N, the lines will connect a vertex X on one bounding line to N − X on the other.

Consider, now, a triangle bounded by the two bounding lines and one of the lines from X to N − X (drawn for X = 4, N − X = 2). The diagonal divides this into 2 triangles. Let us label the distance from the vertex to the intersection with the diagonal as Y_x. If the angle at the vertex between the two bounding lines is θ, we can find the areas of the two triangles using the sine angle formula as

$$A_1 = XY_x sin(\tfrac{\theta}{2})$$
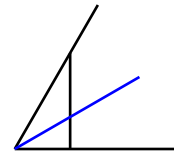
$$A_2 = (N - X)Y_x sin(\tfrac{\theta}{2}).$$ We can find the total area in the same way as

$$A_1 + A_2 = X(N - X)sin(\theta). \text{ So}$$

$$XY_x sin(\tfrac{\theta}{2}) + (N - X)Y_x sin(\tfrac{\theta}{2}) = X(N - X)sin(\theta)$$

$$\implies Y_x = \frac{X(N-X)}{X+(N-X)} \frac{sin(\theta)}{sin(\tfrac{\theta}{2})}$$

$$= X(N - X) \frac{sin(\theta)}{N sin(\tfrac{\theta}{2})}$$

Since we care only about the relative ratios of Y_i we can ignore the constant multiplicative factor and state that $Y_x = X(N-X)$. We now want to re-express those lengths as distances relative to the perpendicular intersection point. We can easily see that this intersection occurs for a line with

$$X = N - X \implies X = \tfrac{N}{2} \text{ so let}$$

$$Z_i = Y_{N/2} - Y_{N/2+i}$$

$$= \tfrac{N^2}{4} - \left(\tfrac{N}{2} + i\right)\left(N - \tfrac{N}{2} - i\right)$$

$$= \tfrac{N^2}{4} - \left(\left(\tfrac{N}{2}\right)^2 - i^2\right)$$

$$= i^2$$

exactly as desired.