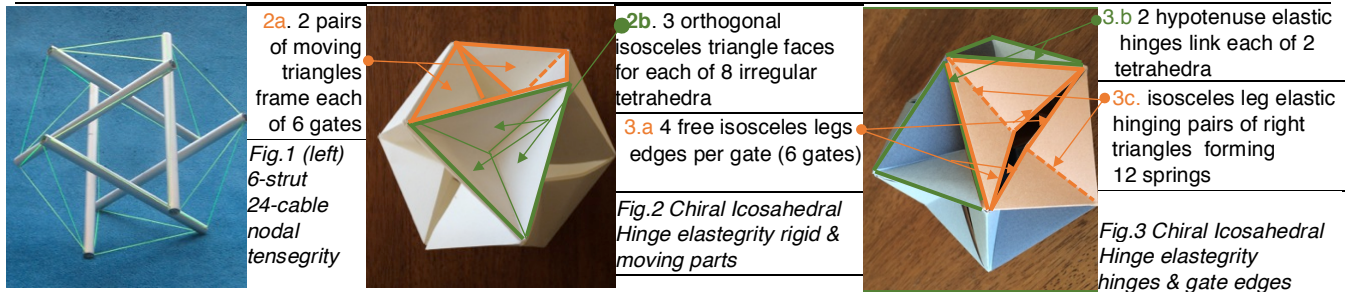


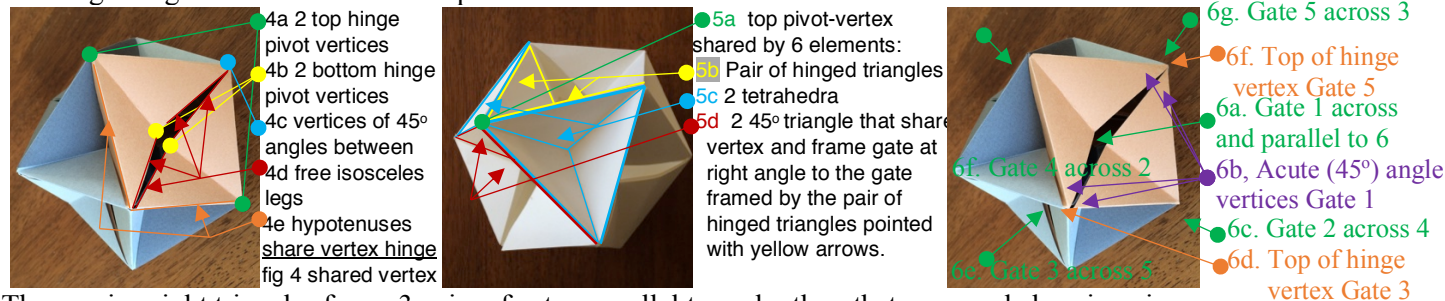
# Chiral Icosahedral Hinge Elastegrity's Geometry of Motion

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## Introduction Hinge Elastegrity, Definitions and Transformations Presented at G4G12



The object that gave rise to the math in this paper is “hinge elastegrities”, a class of structures that originally arose from two Bauhaus exercises assigned at the Yale School of Architecture in the 1970’s and investigated in a series of art projects. The key new object obtained in 1982 involved cutting slits into folded pieces of paper and weaving them into 8 irregular tetrahedra, each with 3 isosceles right-triangle faces outlining an equilateral face fig.2b The 8 tetrahedra are suspended with 12 pairs of moving isosceles-right-triangles, congruent to the tetrahedral face right triangles fig.2a giving rise to an icosahedral shape (not necessarily regular) fig.2. Each pair of right triangles is attached to each other with an elastic hinge, along one of its isosceles legs fig.2a that act as springs. The other isosceles legs are free and frame one of 6, four-sided gates, that open and close as the structure moves fig.3a. The pair of moving right triangles is also attached along their hypotenuse, with elastic hinges, to the hypotenuse of the faces of 2 tetrahedra. The 2 tetrahedra are thus linked by a hinged pair of moving triangles fig.3b. As the structure moves, the dihedral angles between the 12 pairs of moving triangles contract to 0° and expand to 180° while the 24 dihedral angles between tetrahedra and moving triangles contract to 0° and expand to 53.735°.



The moving right triangles frame 3 pairs of gates, parallel to each other, that open and close in unison. Each gate has four sides, defined with four vertices: 2 bottom hinge pivot-vertices fig.4b and 2 acute isosceles angle pivot-vertices formed by the hypotenuses and the free isosceles leg of the moving triangles. fig.4c. The 12 pivot-vertices-of-acute-isosceles-angle are also the same vertices of the 12 top-of-hinge-pivot-vertices linking two adjacent moving triangles fig.4c. that frame an adjacent gate, oriented at right angle to the original gate. For example the top-of-the-hinge pivot-vertex for gate 5 fig.6g is also the pivot vertex of the acute isosceles angle for gate 1 fig.6f, which is at right angles with gate 5. The top of hinge for gate 3 fig.6d is also the pivot vertex for the acute isosceles angle of gate 1 fig.6d. Gate 5 is parallel to gate 3 and both are perpendicular to gate 1.

Each gate is framed with 2 pairs of hinged triangles fig.2a. Each tetrahedron is attached to one of the triangles of 3 hinged triangles pairs fig.7 & fig.8. The 3 elastically hinged triangle pairs, act like springs supporting and linking each tetrahedron with a pair of triangles acting as springs, to 3 tetrahedra. When the structure moves, the 3 tetrahedra rotate with opposite chirality to the original, “floating” like rigid islands in a sea of elasticity to paraphrase Julian Rimoly’s of Georgia Tech, definition for tensegrities. Tensegrities are related structures made of struts, nodally connected with prestressed cables fig 1.

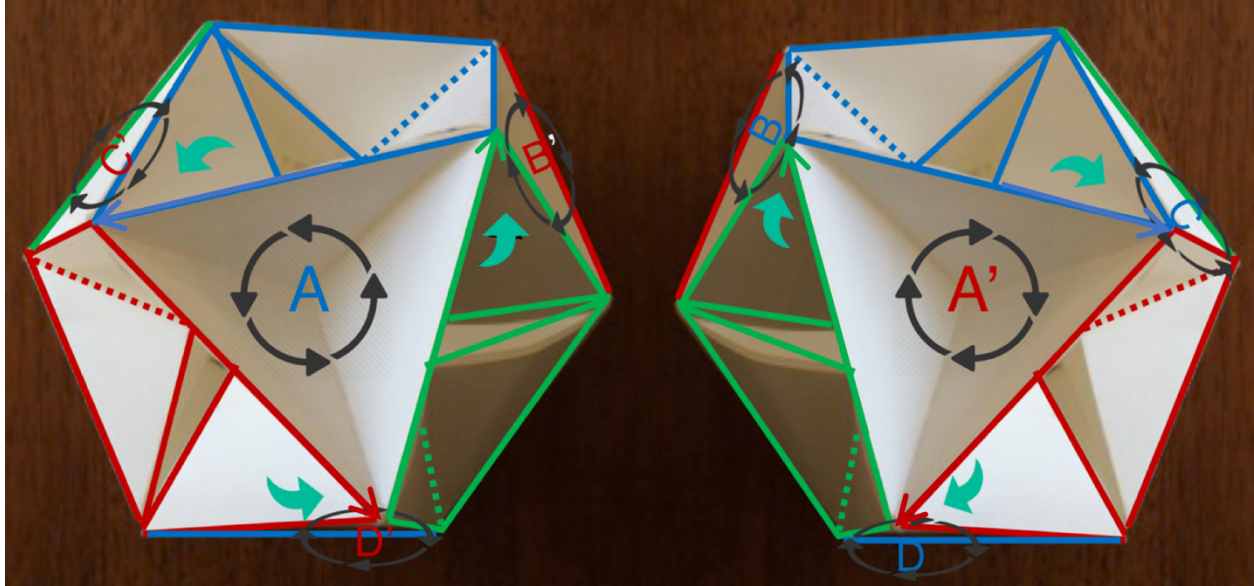


Fig.7 Looking down on the top

Fig.8 Looking up from the bottom

The chiral icosahedral hinge elastegridy has noteworthy physical and geometric properties. When the 8 tetrahedra with 3 orthogonal faces, are compressed along any one of 4 axes, the hinges contract in unison, gyrating 4 tetrahedra clockwise and 4 counterclockwise, until the 8 tetrahedra rest back to back into a regular octahedron. When pulled along one of four axes, the structure extends with reverse gyration into a cuboctahedron. When external forces are removed, elastic forces in the hinges return the structure isometrically, into its original icosahedral shape (not necessarily regular). Because of similarities<sup>i</sup> in symmetry and elasticity of the structure with tensegrity figures Fig. 1 that maintain shape integrity by pre-stress tension alone, these new objects that maintain shape integrity through elastic hinges<sup>ii</sup> were named “hinge elastegridy”.

The hinge elastegridy’s shape-shifting through further folding was presented at G4G12 and it led to a number of familiar geometric objects, as well as some new ones fig. 9. The hinge elastegridy can flatten into a multiply covered square, morph into shapes with the vertices of each of the Platonic shapes, model the hypercube, transform into objects symmetrical to 6-strut, 12-strut, 30 strut, and 60 strut tensegrity, as well as take the shape of new figures with the vertices of congruent faces that are not regular polygonal regions. Co-presenter at G4G12, professor Thomas Banchoff, termed these figures monohedra. The object obtained through folding and trigonometry, has twelve congruent pentagons, each pentagon having one right angle. The side of the pentagon across the right angle is 0.54.. of the four equal sides. Using analytic geometry, Banchoff generalized this unique new monohedron-dodecahedron into a continuum family of monododecahedra (f=12 congruent not regular). Through folding, the smaller side can vary from expanding to be equal to the other 4 sides becoming regular dodecahedron, to decreasing to 0, the variable side becoming a point, thus the monohedron dodecahedron becoming rhombic.

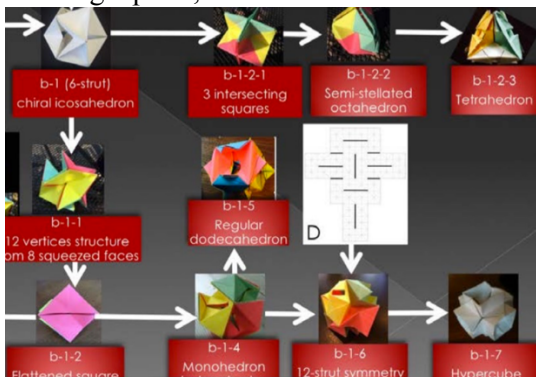


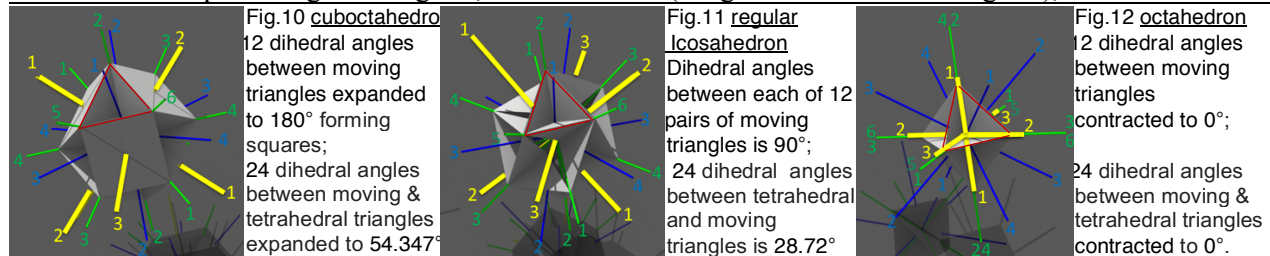
Fig. 9 Partial Elastegridy tree of forms through folding  
**Platonic Solids:** Tetrahedron **b-1-2-3**, Cube **b-1-3**, Octahedron **b-2-2**, Regular dodecahedron **b-1-4**, Icosahedron **b-1**;  
**Tensegrities:** Same symmetry as a 6-strut **b-1**; 12-strut **b-1-6**  
**Jitterbugs gyration symmetrical in 2 directions along 3 axes:** Cube **b-1-3**, Flat square goes up and down, **b-1-2**;  
**Mono-dodecahedron continuum**  
 Pentagonal dodecahedron 1 right angle **b-1-4**, => regular **b-1-5** => rectangle with one angle 180°  
**2-d Square: b-1-2, to 3-d Cube b-1-3, to 4-d Hypercube: b-1-7.**

**D** fractal of **B** derived through 3-d folding of **b-1** into **b-1-1**, into **b-1-3** (they have the same creases but different orientation in 3-d)

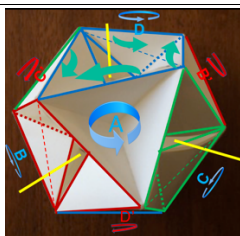
## Geometry of Motion of 13 axes and Movement of Vertices Outlining a Dodecahedron

The geometry of motion of the Chiral Icosahedral Hinge Elastegrity's members is presented here, in relation to 13 axes in 3 sets: a. a set of 4 tetrahedral, fig. 5, 6, 7 blue, b. a set of 3 orthogonal, fig. 5, 6, 7 yellow, and c. a set of 6 diametric axes through opposite icosahedral vertices, fig. 5, 6, 7 green.

Also we present the geometry resulting from movement of 20 vertices: 12 hinge-bottom-pivot-vertices together with 8 right-angles-vertices of the orthogonal tetrahedral faces, outline a dodecahedron. We show that a force applied on the icosahedron actuates motion of the 20 vertices replicating through movement the same geometric transformations, as those presented at G4G12 through folding: a continuum of dodecahedra shapeshifting from regular, to monohehedral (congruent faces but not regular), to rhombic.



### a. Tetrahedra translate and spin along 4 axes (blue)



4 tetrahedral axes through the center of opposite tetrahedra A-A', B-B', C-C', D-D'  
3 orthogonal axes through the center of the gates showing,

- counterclockwise spinning of A, B, C, D
- clockwise spinning of A', B', C', D'

Tetrahedra D and B' share common pivot vertex DB' one on one side of the gate and tetrahedra A and C' share common pivot vertex AC' on the other side of the gate and rotate around their shared hinges so that common vertex C'D and AB' become congruent when the structure contracts into an octahedron.

Fig. 13 Clockwise and counterclockwise tetrahedral spinning around 4 axes as 6 gates close

When a pair of diametrically placed tetrahedra are pressed together, along any of 4 axes, defined by opposite centers of equilateral faces of the 8 tetrahedra (blue fig.5, 6, 7), the 4 axes gyrate around the structure's center fig.13. The 8 asymmetrical tetrahedra spin and slide along the gyrating axes towards the center, into an octahedron. As the structure contracts, the orthogonal faces of 2 tetrahedra sharing a pivot vertex rest back to back, squeezing between them, a folded pair of hinged moving right triangles, also hinged to the 2 tetrahedra along their hypotenuse. As the 6 gates close, the 8 right angle vertices of the orthogonal tetrahedral faces and the 12 bottom hinge pivot points, become congruent with each other and with the center of the structure.

When any two opposite tetrahedra are pulled away from each other, along any of the 4 axes, the 36 elastic hinges also actuate simultaneously movement of the entire structure. The 8 tetrahedra spin and slide away from the structure's center, in unison along the 4 axes, as the axes gyrate with reverse chirality, around the structure's center. The 12 dihedral angles of moving pairs of triangle, open to  $180^\circ$ , while the 24 hypotenuse dihedral angles expand to  $54.375^\circ$ . The 8 equilateral triangle faces of the tetrahedra rotate as they move away from the center to become the 8 equilateral faces of a cuboctahedron fig.10 red triangle.

### b. Movement of the gates around 3 stable orthogonal axes (yellow)

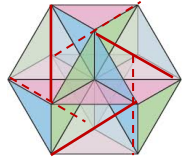
3 orthogonal axes are defined by the centers of the 3 opposite pairs of gates. When a force actuates motion along any of the tetrahedral axes activating contraction or expansion of the 36 hinges, the 6 gates axes open and close around the 3 axes. The centers of the gates slide away in 8 directions when the structure is expanding, and towards the center when the structure is contracting.

As the dihedral angles of the moving triangles approach  $0^\circ$ , the angle between the free isosceles legs pivoting around the bottom-hinge-vertex also approach  $0^\circ$ . Simultaneously the width of the gates, the distance between the two bottom-hinge-pivot-vertices across each gate, decrease approaching zero.



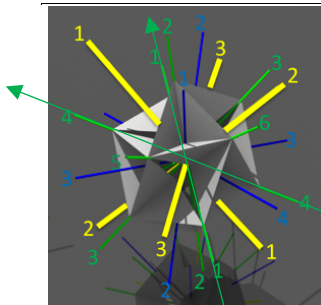
As the 36 dihedral angles contract to  $0^\circ$  and the structure contracts into an octahedron, each of the 6 sets of 4 rotated isosceles legs, edging the gates, become congruent with the axes of the octahedron. The orthogonal axes of the structure become congruent with the orthogonal axes of opposite vertices of the octahedron. In addition a) the 2 hinges linking the 4 triangles surrounding each gate, b) the 4 edges of the 2 pairs of tetrahedra that each shares the top of the 2 hinge vertices of a gate on either side fig.4a. as well as c) the 4 gate-edges contracted to  $0^\circ$ , all 10 (2 hinges+4 tetrahedral edges+4gate-edges) become congruent.

As tension is applied on any of the 4 tetrahedral axes, the gates open pivoting with opposite chirality around the 3 axes. As the dihedral angles of the moving triangles open towards  $180^\circ$ , the gates' width decreases again. When the dihedral angle becomes  $180^\circ$ , the two moving right triangles on either side of the hinges, align into 2 bigger right triangle, on either side of the gate, flattening into a square Fig.14. The free isosceles legs that are the gate edges, rotate  $180^\circ$ , closing the gates, and forming the diagonal of the square faces of the cuboctahedron.  
 Fig.14 Diagonals of cuboctahedron square faces show closed gates when structure is expanded.



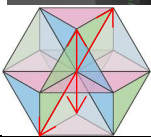
The 3 orthogonal axes pass through the centers of the 6 diagonals of the cuboctahedron squares that are the closed gates of the structure. Somewhere between the gate-edges closing by rotating the angle between them to  $0^\circ$ , when they align becoming congruent with the orthogonal axes and each other, and closing by rotating to  $180^\circ$ , expanding into the diagonals cuboctahedron square faces, the opening of the gates' becomes maximum. As we will discuss with more detail below the maximum gate width is achieved when the dihedral angles between moving triangles is  $90^\circ$ .

### c. 6 axes passing through diametrically opposite icosahedral vertices



A set of 6 axes (green) is defined by pairs of diametrically opposite vertices of the icosahedron, forming 3 pairs of X. Each X passes through the 4 acute-angle-pivot-vertices of two parallel gates on either sides of the structure's center Fig.4b. In the illustration Fig.15 the 3 pairs of X are formed by axes 4 & 2, 3 & 6, 5 & 1. As the structure contracts the 3 pairs of X pivot towards each other, as they close around the 3 orthogonal axes, until each pair of axes becomes congruent with the orthogonal axes that pass through the gate centers 4 & 2 with axis 1, 3 & 6 with axis 2, and 5 & 1 with axis 3.

Fig 15 Diametric axes 4 & 2, 1 & 3, 5 & 1 form X, congruent with 3 orthogonal axes when closed.



When the structure expands into a cuboctahedron, each pair of axes passing through diametrically opposite vertices forming an X open to  $60^\circ$ .

Fig 16 When structure expands into a cuboctahedron 3 X each open to  $60^\circ$

### d. Regular dodecahedron, to monohedron, to rhombic dodecahedron through movement

Examining the movement of the 13 axes, and in particular pondering the closing, opening, and closing again of the gates, as the structure contracts into an octahedron and expands into a cuboctahedron, raised the question, at what point is the gate width maximal?

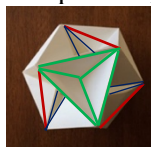


Fig.17 dihedral angle =  $90^\circ$

Attempting to answer the question of when do the gates achieve maximal width led to the realization that when the distance between 2 vertices across a gate is equal (fig.17 red line) then the implied 12 tetrahedra, created by drawing the 6 red lines, are congruent with the 8 tetrahedra suspended by the moving triangles (fig.17 green lines) and therefore the dihedral angle between each pair of moving triangles is  $90^\circ$ . The distance from each of the 20 right angle vertices of the orthogonal faces of the 20 congruent tetrahedra to the 3 adjacent right angle vertices of the 20 congruent tetrahedra is equal and the 20 vertices outline a regular dodecahedron. 2 of the vertices shown with black dots fig.18 show the width of a gate, which is equal to the 30 edges the regular dodecahedron shown on fig.19. In fig.18 the 8 "floating" tetrahedra are A, B, C, D and diametrically opposite with reverse chirality A', B', C', D'. In fig.19 the bottom of hinge pivot vertices are indicated with the letter of the two tetrahedra sharing the hinge: AB', AC', AD', BC', BD', CD', A'B, A'C, A'D, B'C, B'D, C'D, the right angle vertices of the orthogonal faces of the 8 floating tetrahedra are indicated with A, B, C, D, A', B', C', D'

$$a_{\text{dodecahedron}} = r_{\text{u dodecahedron}} \text{ (radius of a circumscribed sphere in a dodecahedron)} / 1.401258538 \dots \Rightarrow a_{\text{dod}} = 0.248a_{\text{ico}} \text{ (derivation below)}$$

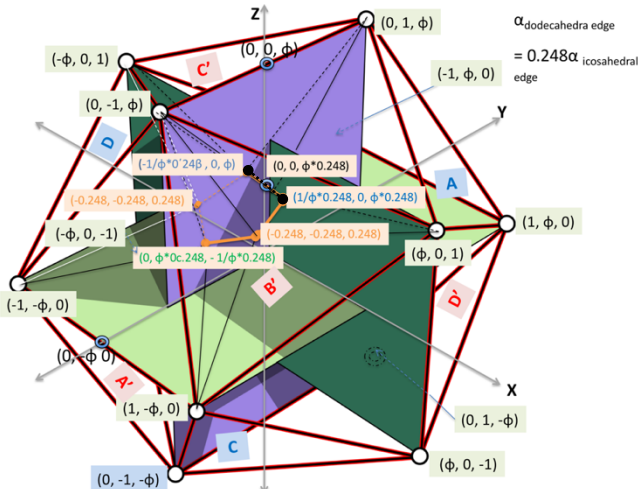


fig.18 Coordinates of a regular icosahedron when The dihedral angles are  $90^\circ$

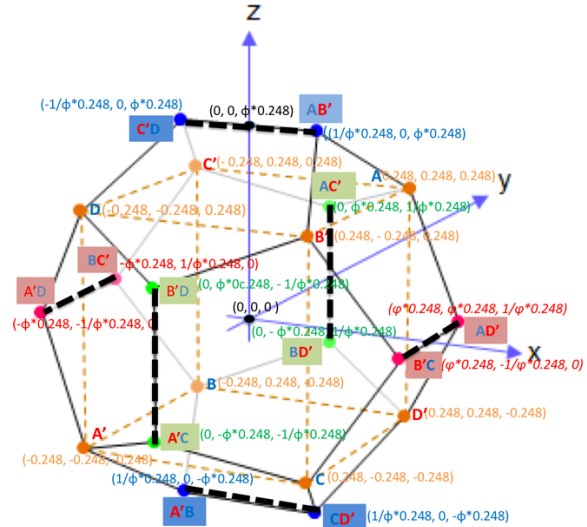


fig.19 Coordinates of regular dodecahedron when the dihedral angles are  $90^\circ$ , pentagonal sides are equal, dotted lines indicated gates.

$r_{u\text{ dod}} = r_{i\text{ ico}}$  (radius of inscribed icosahedron)  $- h$  height of tetrahedron  $\Rightarrow 0.7558a_{\text{icosa}} - 0.408a_{\text{icosa}} = 0.348a_{\text{icosa}}$

- because  $r_{\text{icosahedron}}$  (radius of inscribed icosahedron)  $= 0.7557623142a_{\text{icosa}}$
- and  $h_{\text{tetrahedron}} = r_{\text{octa}}$  (radius of inscribed octahedron)  $= r_{\text{octacoron}} = 0.408a_{\text{octa}} = 0.408a_{\text{ico}}$  ( $a_{\text{octa}} = a_{\text{icosa}}$ )

so  $r_{u\text{ dod}} / 1.401258538 = 0.248a_{\text{icosahedron}}$  is the width of the gate when the icosahedron is regular.

1. When a force contracts the icosahedron, so that the moving triangle hinge dihedral angles is less than  $90^\circ$ , then the 20-vertex-dodecahedron decreases isometrically and proportionally, and remaining regular, until the 20 vertices become congruent with the center of the structure, and shrink to a point when the structure contracts into an octahedron.
2. When a force expands the icosahedron so that the moving triangle dihedral angles are greater than  $90^\circ$ , then the width of the 6 gates decrease again while the other 4 pentagonal sides increase.

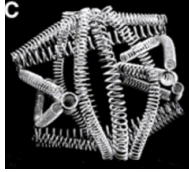
**Therefore 6 gate Max width =  $0.248a_{\text{ico}}$  when the dihedral angles between moving triangles =  $90^\circ$**

- A polyhedron with congruent non-regular faces, (4 equal pentagonal sides but one side smaller) has been termed a monododecahedron, in G4G12. With professor Banchoff we presented a similar transformation of the chiral icosahedral hinge elastegritty through folding giving rise to a continuum of pentagonal monododecahedra, until the decreasing side = 0, when the dodecahedron is transformed to rhombic.
- When the dihedral angle expands to  $180^\circ$  the gate width is 0, the 2 bottom of the hinge vertices on either side of each gate become congruent, reducing the number of vertices from 12 to 6. Together with the 8 right angle vertices of the orthogonal faces of the tetrahedra, the dodecahedron is a 14 vertex rhombic.

## From a physics point of view and applications (experimentally derived)

- When the stiffness of a shape-memory membrane creates an elastegritty with the dihedral angles sagging below  $90^\circ$  (or when the structure starts contracted with  $0^\circ$  dihedral angles), a matrix of chiral icosahedral hinge elastegritties behaves as a tensile spring.
- When the a shape-memory membrane is stiff and the 12 dihedral angles are greater than  $90^\circ$  (or start expanded to  $180^\circ$ ), a matrix of chiral icosahedral hinge elastegritties acts as a compression spring.

There are several possible applications for the Chiral Icosahedral Hinge Elastegritty cited in a footnote in the paper for G4G12, including a number of existing tensegrity applications that may be improved with the additional properties of hinge elastegritties and some novel applications specific to the additional unique properties of hinge elastegritties.



One of the applications proposed for tensegrities, is Donald Ingber's, conjecture that all biological structure is hierarchically ordered tensegrities. Ingber is a founding director of the Wyss Institute for biological engineering and has been publishing on this topic for over 35 years. Ingber proposed a model of tensegrities where compression coil springs take the place of struts and tensile coil springs take the place of cables. This model can be

Fig. 20 Ingber Model also seen in his most recent publication *Multi-scale modeling reveals use of hierarchical tensegrity principles at the molecular, multi-molecular, and cellular levels* C. Reilly, D. Ingber, *Extreme Mechanics Letters* 20, (2018) 21. In this most recent article Ingber proposes a force and energy distribution

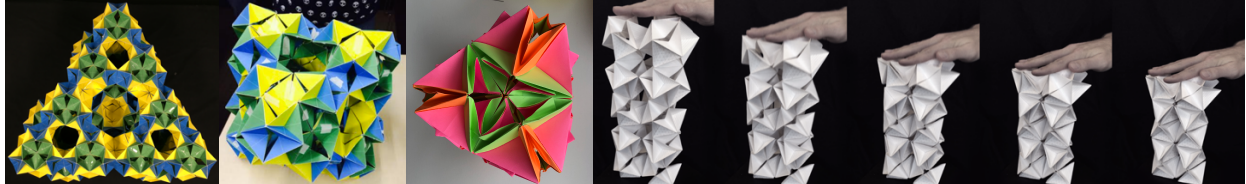


Fig.21 a. & b. dihedral angle > 90° b. dihedral angle < 90° c. NPR (Negative Poisson's Ratio gets narrower as it gets shorter)

argument. Hinge elastegrities address several questions that tensegrities leave unanswered including accounting for the fact that numerous independent biological papers expressed astonishment in measuring experimentally NPR in different parts of the anatomy of various species, suggesting that Negative Poisson's Ratio is ubiquitous in all of the architecture of life. For links to these articles please email to request. Additional elastegrities suggest the design of pumps for non-Newtonian fluids, that are prevalent in biological structure, and self-assemble into a structures at hierarchically different scales with smaller scale elastegritiy components through folding a shape memory membrane.

If you are interested to collaborate in investigating the force distribution and energy transmission of a matrix of Chiral Icosahedral Hinge Elastegrities fig.21 please conduct me at [epavlidis@RWU.edu](mailto:epavlidis@RWU.edu). It may open the gate to numerous applications that you may be interested to develop collaboratively.

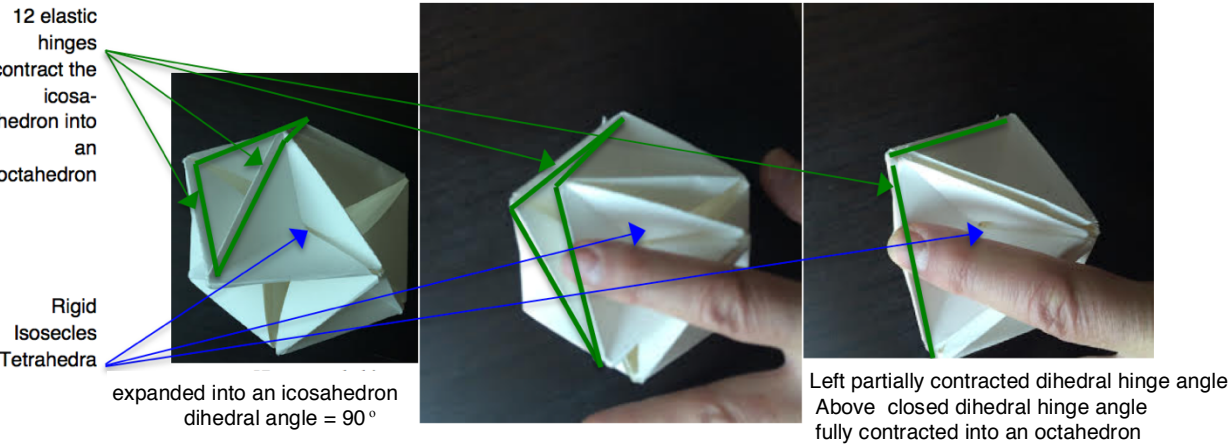


Fig.22 Chiral Icosahedral Hinge Elastegritiy contracting into an octahedron

- i Structurally both tensegrities and elastegrities are networks of rigid (linear struts for tensegrities, irregular tetrahedra for the icosahedral hinge elastegritiy) and elastic (pre-stressed cables acting in pure tension for tensegrity and elastic hinges for elastegrities)
- ii The term "hinge" differentiates elastegrities from those that may be termed "nodal". Nodal elastegrities can be created by replacing tensegrity's pre-stressed cables, holding together struts with springs. Additional hinge elastegrities have been created from further folding, weaving, and inverting the original icosahedron. One may consider elastegrities both "nodal" and "hinge" as the general family of structures that tensegrities are a special subcase