One Puzzle

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1. The Broken Calculator

A calculator is missing all of its keys but sin, cos, tan, SHIFT[1] and =. It initially starts with 0 on screen. Show that the calculator can produce any positive rational number.

[1] That is to say: the inverse trigonometric functions are also available.

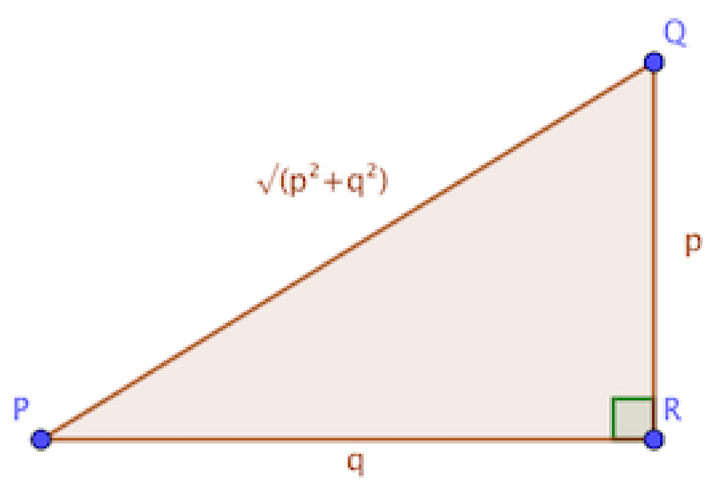
Some functions

By applying one of the three inverse functions to a number (assuming it is in the relevant domain) and one of the direct functions to the result, we end up with a (generally different) number. It's worth exploring some of the things we can do with such compositions.

A useful composition would be one that took a number greater than one and returned its inverse, so that the output is in the domain of all three inverse functions.

This can be arranged by considering a right-angled triangle as pictured, with $q > p$. $\arctan\left(\frac{q}{p}\right)$ gives angle Q. The cosine of Q is $\frac{p}{\sqrt{p^2+q^2}}$, and the arcsine of this is angle P. Finally, $\tan(P) = \frac{p}{q}$, the reciprocal of the original argument.

Figure 1: A triangle



Definition: Let $R(x) = \tan\left(\arcsin\left(\cos\left(\arctan\left(x\right)\right)\right)\right) = \frac{1}{x}$.

Given a number smaller than one, where do the various compositions leave us? Ignoring the self-inverse compositions, and assuming $p < q$, we have:

- $\sin\left(\arccos\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{q}$
- $\cos\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{q}$
- $\tan\left(\arccos\left(\frac{p}{q}\right)\right) = \frac{\sqrt{q^2-p^2}}{p}$
- $\cos\left(\arctan\left(\frac{p}{q}\right)\right) = \frac{q}{\sqrt{p^2+q^2}}$
- $\tan\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{q^2-p^2}}$
- $\sin\left(\arctan\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{p^2+q^2}}$

I've arranged these in three pairs, such that each element of a pair is the other's inverse over a domain of at least $0 \le \frac{p}{q} \le 1$.

The first pair of functions aren't especially interesting, but either of the last two pairs can be used to great effect. I'll pick the last pair, and give them names.

Definition: Let $T_s(x) = \tan\left(\arcsin\left(x\right)\right)$.

Definition: Let $S_t(x) = \sin\left(\arctan\left(x\right)\right)$.

With these two functions, and $R(x)$ from before, we can solve the puzzle.

A solution

Proposition: Any positive rational number can be produced by applying a composition of the functions sin, cos, tan and their usual restricted inverses to 0.

Remark: $\cos(0) = 1$, so 1 can be produced.

Demonstration: Suppose we wish to produce a rational number, $r = \frac{p}{q}$, with p and q coprime positive integers.

If $p > q$, then r can be produced if $\frac{q}{p}$ can; therefore, we need only show that all positive rational numbers smaller than 1 can be reached.

Assuming $r < 1$, it can be reached (by way of S_t) if $T_s(r) = \frac{p}{\sqrt{q^2-p^2}}$ can.

This is not (generally) a rational number, but it is the square root of a rational number. Its numerator is smaller than q, by supposition; its denominator is also smaller than q because of geometry and/or algebra[2].

Remark: The key point here is that $T_s\left(\frac{p}{q}\right)$ is a fraction with a numerator and denominator both of which are square roots of integers, and both strictly smaller than q.

Applying R if needed, this means $\frac{p}{q}$ can be generated from some number of the form $\frac{\sqrt{a}}{\sqrt{b}}$ with $1 \leq a \leq b < q$, with a, b and q all integers[3].

Repeating the process leads to still smaller elements of the fraction; a decreasing sequence of integers bounded inclusively from below by 1 must eventually reach 1.

Since we know we can produce 1, all positive rational numbers can be produced $\blacksquare$.

[2] You can see this by considering a right-angled triangle with hypotenuse q and a leg of p. The second leg is $\sqrt{q^2 - p^2}$, which is smaller than q.

[3] Regarding $a \leq b$: equality holds here iff $q = \frac{1}{\sqrt{2}}$.

An example

Suppose we want to produce $r = \frac{4}{3}$, everyone's favourite triangle-related fraction.

- $\frac{4}{3}$ can be produced if $\frac{3}{4}$ can; $r = R\left(\frac{3}{4}\right)$.
- $\frac{3}{4}$ can be produced if $\frac{3}{\sqrt{7}}$ can: $r = R\left(S_t\left(\frac{3}{\sqrt{7}}\right)\right)$.
- $\frac{3}{\sqrt{7}}$ can be produced if $\frac{\sqrt{7}}{3}$ can: $r = R\left(S_t\left(R\left(\frac{\sqrt{7}}{3}\right)\right)\right)$.
- $\frac{\sqrt{7}}{3}$ can be produced if $\frac{\sqrt{7}}{\sqrt{2}}$ can: $r = R\left(S_t\left(R\left(S_t\left(\frac{\sqrt{7}}{\sqrt{2}}\right)\right)\right)\right)$.
- $\frac{\sqrt{7}}{\sqrt{2}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{7}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(\frac{\sqrt{2}}{\sqrt{7}}\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{7}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{5}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(\frac{\sqrt{2}}{\sqrt{5}}\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{5}}$ can be produced if $\frac{\sqrt{2}}{\sqrt{3}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{\sqrt{3}}$ can be produced if $\frac{\sqrt{2}}{1}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(\frac{\sqrt{2}}{1}\right)\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{\sqrt{2}}{1}$ can be produced if $\frac{1}{\sqrt{2}}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(\frac{1}{\sqrt{2}}\right)\right)\right)\right)\right)\right)\right)\right)\right)$.
- $\frac{1}{\sqrt{2}}$ can be produced if $\frac{1}{1}$ can: $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(S_t\left(\frac{1}{1}\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)$.
- $1 = \arccos(0)$, so $r = R\left(S_t\left(R\left(S_t\left(R\left(S_t\left(S_t\left(S_t\left(R\left(S_t\left(\arccos(0)\right)\right)\right)\right)\right)\right)\right)\right)\right)\right)$.

Therefore $\frac{4}{3}$ can be produced.

A connection

"Why are you writing all this, Colin? It's a diverting enough puzzle, but… why?"

I'm writing about it because it gave me such a lovely revelation, I nearly jumped out of the bath.

Suppose we write our target fraction as $r = \frac{\sqrt{P}}{\sqrt{Q}}$, with $P = p^2$ and $Q = q^2$. Then our algorithm for working backwards to show 1 can be produced from r (and, hence, by way of inverses, r from 1) is:

While $Q \neq P$:

- If $Q < P$, swap P and Q (this is the effect of $R\left(\frac{q}{p}\right)$).
- Let $Q = Q - P$ (this is the effect of $T_s\left(\frac{p}{q}\right)$).

This is Euclid's algorithm for finding the greatest common factor of P and Q! Since, by supposition, P and Q are coprime, their GCF is 1. Therefore, 1 can be produced from r and hence r can be produced from 1 ■.