

*Gathering for Gardeners:
A Randomized Approach to Pattern Formation*

*Shunhao Oh and Dana Randall
Georgia Institute of Technology*

As any gardener can attest, ants love to congregate. They relentlessly gather around crumbs, form robust trails, and carry out a variety of simple, coordinated tasks that address the colony's current needs. But how do they coordinate when no single ant is in charge, their communication is limited, and the necessary task may change depending on external conditions known only to a select few? Somehow, ants seem to know exactly when a ripened piece of fruit drops to the ground and, much to the gardener's chagrin, waste no time gathering in a coordinated feeding frenzy. On the other hand, once the food is depleted or they are chased off by an angry gardener, they quickly disperse, all simultaneously executing a new protocol to forage or to run for their lives.

While it is difficult to know exactly what clever ants can accomplish, or how they coordinate so effectively, one can try to model ant-like behaviors with dumbed down *self-organizing particle systems*, where particles, rather than ants, interact via very rudimentary instructions. The question of what can be computed in such a computationally limited, distributed setting is especially compelling because many engineered, physical, and social sciences contain collectives known to self-organize.

Gathering (or *aggregation*) is an illustrative example. How can a system of homogeneous particles, with no global orientation or communication, be made to *aggregate*, forming tight-knit communities, or *disperse*, the inverse action where they spread out and explore? Aggregation and dispersion protocols are found in many natural systems, such as fire ants gathering to form rafts [1] and honey bees communicating foraging patterns by swarming closely within their hives [4]. While each individual ant or bee lacks global knowledge of the collective, it can take cues from its immediate neighbors to achieve global coordination. Similarly, systems of heterogeneous (say, colored) particles can self-organize into either *separated* (or segregated) and *integrated* states, depending on what is most advantageous to the group based on external circumstances. Examples of separation include molecules exhibiting attractive and repulsive forces, strains of bacteria competing for resources while also collaborating to-

wards common goals [6, 7], and social insects acting belligerently or friendly towards other colonies when threats are introduced or removed [5].

A goal for understanding collective behaviors is to find distributed, local algorithms that, when run by each particle independently and concurrently, result in emergent self-organization such as *separation* or *integration* of color classes. In [2] and [3], we presented simple stochastic, distributed algorithms that *provably achieve aggregation/dispersion and separation/integration* by adjusting just a couple of parameters slightly that control each particle's affinity for other nearest neighbors or nearest neighbors of the same color. Adjusting these parameters causes the entire system to undergo a system-wide *phase change*. Thus, each of these collective behaviors can be viewed as *emergent global outcomes of local interactions*, much like phase transitions that turn water into ice or spontaneously magnetize a metal below critical temperatures.

Taking inspiration from these models of biological and physical systems, we now ask what happens if the particles are even more particular about what types of neighbors they prefer. Rather than simply preferring more neighbors, or more like-colored neighbors, what if the particles strongly prefer to have exactly 4 neighbors? What if they prefer 3 red and 3 blue neighbors? Should we expect more phase changes where particles are disordered below some threshold and begin to form long-range organization above some other threshold, much as we see in many particle systems studied in statistical physics?

We explore such questions here with particles that are red or blue and reside on some finite region of the triangular lattice (we add toroidal boundary conditions by identifying left and right sides, as well as top and bottom sides, of a large rhomboidal region so that every vertex on the lattice region has exactly six neighbors). Each vertex is then occupied by a red or blue particle, and particles can swap places, with each trying to find a location where its neighbors have the color ratios they most prefer.

As expected, striking patterns emerge! This is not entirely surprising because we can view red and blue particles as species of ants, and if each prefer six same colored neighbors to five, and five is favored over four, and so forth then we are mimicking the separation algorithm that is known to gather the particles of each color class together. We demonstrate that using other types of *neighbor-aware particles* that favor exactly 3 neighbors sharing their color, or 4, we can get striking patterns of global coordination. What is even more intriguing (to us!) is that the emergent patterns are highly dependent on the density of red particles in the mixture. In addition to long-range order emerging when local affinities are strong enough, we also find remarkable phase changes among emergent patterns as the density is increased, with stripes morphing to polka dots starting locally and spreading over the entire region. Here we demonstrate this behavior with simulations and conjectures.

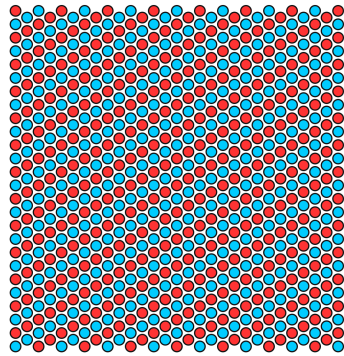
Neighbor-Aware Particles

Let's imagine red and blue colored particles fully occupying all the vertices of a region on the triangular lattice with sites and toroidal (or periodic) boundary conditions. We call this region $G_\Delta = (V, E)$, where V are the vertices and E are the edges. We are going to fix the proportion $\rho = N_{red}/N$ of red particles, where $|V| = N = N_{red} + N_{blue}$, the number of particles of each color. The particles know their own color and the colors of each of their immediate six neighbors, and each particle knows its *homophily preference*, i.e., what ratios of like and unlike colors it prefers in its immediate neighborhood.

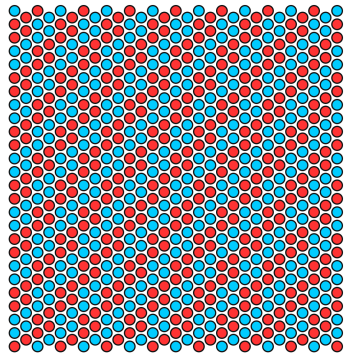
First, consider what the configurations will look like if we try to maximize the number of particles that achieve their homophily preferences. As an example, consider a region with $\rho = .5$ where all particles have homophily preference of 2, so they want exactly 2 neighbors to have their color and the remaining 4 to have the other color. Figure 1.a shows one way that all vertices can simultaneously achieve this goal. Similarly, when $\rho = .5$ and each particle wants 3 neighbors of each color, then again each particle can satisfy its homophily preference of 3, as shown in Figure 1.b. In Figure 1.c we see how to satisfy every particle's homophily preference when $\rho = 0.5$ and each particle favors 4 neighbors of its own color.

But the striped patterns shown in Figures 1.a, 1.b and 1.c only appear when there are equal numbers of red and blue particles. As we start modifying the density of each, we cannot always make every particle happy and some vertices need to be "sacrificed" to help others. Figures 1.d, 1.e and 1.f show patterns where the maximum number of particles achieving satisfy their homophily preferences. Notice that in Figure 1.d when vertices want 4 like colored neighbors and $\rho = 0.25$, the blue vertices all achieve optimal homophily but none of the red vertices do. We say that this configuration has an *efficiency* ξ of 3/4 because 75% of the vertices satisfy their homophily preferences.

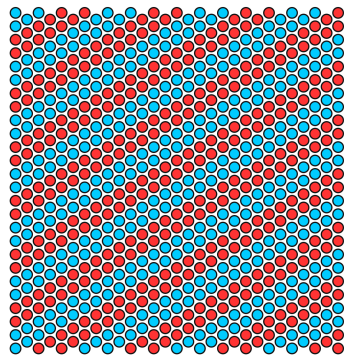
The last two examples in Figure 2 are for homophily preference 3. In Figure 2.e, $\rho = 1/3$ and the efficiency is $\xi = 2/3$ (meaning a third of the particles are red and the efficiency comes from the blue vertices, which all have the desired three blue neighbors). In Figure 2.f, $\rho = 6/13$ and the efficiency is $\xi = 12/13$ (since all vertices except for the centers of the blue hexagons have the desired homophily preference). The first three examples in this figure all achieve efficiency 1 since all vertices have their optimal homophily values.



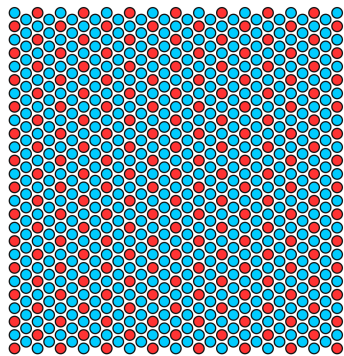
(a) degree 2, $\rho = 0.5$



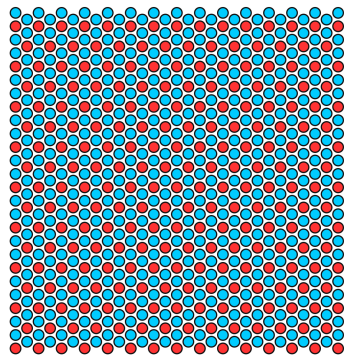
(b) degree 3, $\rho = 0.5$



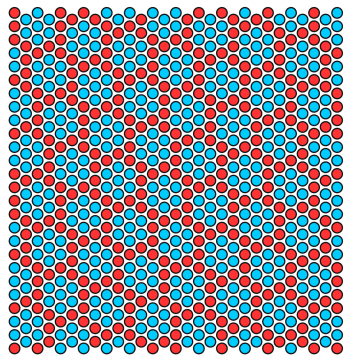
(c) degree 4, $\rho = 0.5$



(d) degree 4, $\rho = 0.25$



(e) degree 3, $\rho = 1/3$



(f) degree 3, $\rho = 6/13$

Figure 1: Maximizing vertices with the desired degree at specified densities.

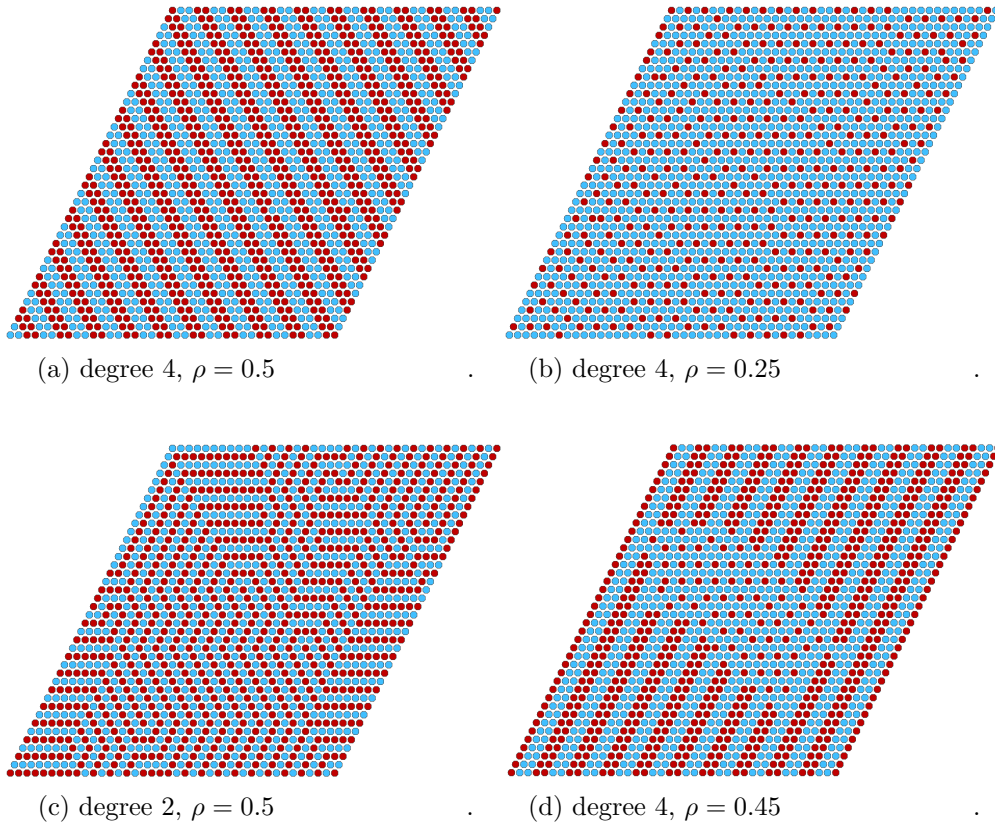


Figure 2: Simulations of the probabilistic model with various homophily preferences and densities.

The Probabilistic Setting

A far more compelling situation arises when we define preferences in terms of a probability distribution that, rather than trying to maximize the number of particles that achieve their homophily preferences, just makes such configurations more likely. To do this, we define the weight of any particular configuration to be the product of the individual particles' satisfaction with the colors of their neighbors. More precisely, fix $\lambda_0, \lambda_1, \dots, \lambda_6$ and for each i from 0 to 6, let $\lambda_i > 0$ be the relative homophily values that a particle derives when exactly i of its neighbors agree with its own color. Let Ω be the set of valid configurations, i.e., those with ρN red vertices and $N - \rho N$ blue vertices. For any configuration $\sigma \in \Omega$, we define its weight as $wt(\sigma) = \prod_{v \in V} \lambda_{s(v)}$, where $s(v)$ is the number of neighbors w of v such that $\sigma(v) = \sigma(w)$. When we normalize this weight by dividing by the sum of the weights of all possible configurations, we turn this

into a probability distribution:

$$\pi(\sigma) = \prod_{v \in V} \lambda_{s(v)} / Z,$$

where

$$Z = \sum_{\tau \in \Omega} \prod_{v \in V} \lambda_{s(v)}.$$

For a homophily preference of 4, for example, we may set $\lambda_4 > 1$ and for all $i \neq 4$, we set $\lambda_i = 1$. Note that as λ_4 gets larger, the distribution starts favoring configurations that have an increasing number of vertices with the desired homophily values.

Figure 2.a shows what happens if we have homophily preference 4 and density $\rho = 0.5$. The orientation of the stripes that emerge can lie in any of three directions and there will always be some defects throughout the pattern that arise randomly. In Figure 2.b we show a homophily preference of 4 and density $\rho = 0.25$. Here we see a grid-like pattern emerging. In Figure 2.c, we have homophily preference 2 with density $\rho = 0.5$. Lines similar to those in Figure 1.a form, but the sporadic degree 3 vertices that arise in the probabilistic setting can cause the lines to curve and wrap around.

In Figure 2.d we see something different. When $\rho = 0.45$, it is not possible to have as many vertices fulfill their homophily preferences as when the density was 0.25 or 0.5. At such intermediate densities, the best one could do is to have part of the region produce a “ $\rho = 0.25$ ” type pattern and part produce a “ $\rho = 0.5$ ” type pattern. This is exactly what emerges when sampling configurations at this intermediate density. Moreover, by nearly minimizing the boundary between these two patterns, we reduce the number of vertices that fail to achieve either nice pattern, and this is also what is observed in Figure 2.d.

We call configurations with patterns that fill the whole region *pure*, such as Figures 2.a and 2.b, and configurations that show multiple patterns simultaneously *mixed*, as in Figure 2.d. Note that since 0.45 is four-fifths the way between 0.25 and 0.5, we expect to see about 4/5 of the region looking like a pure pattern arising from $\rho = 0.25$ and 1/5 looking like the pattern arising from $\rho = 0.5$. In other words, since there is no pure pattern occurring at $\rho = 0.45$, the particle system compromises by optimally mixing the two closest pure patterns in each direction.

Conjectures

Graphically, we can map out what happens for all $\rho \in [0, 1]$ for homophily preference 4 on a diagram. Recall that each “pure pattern” is associated with a

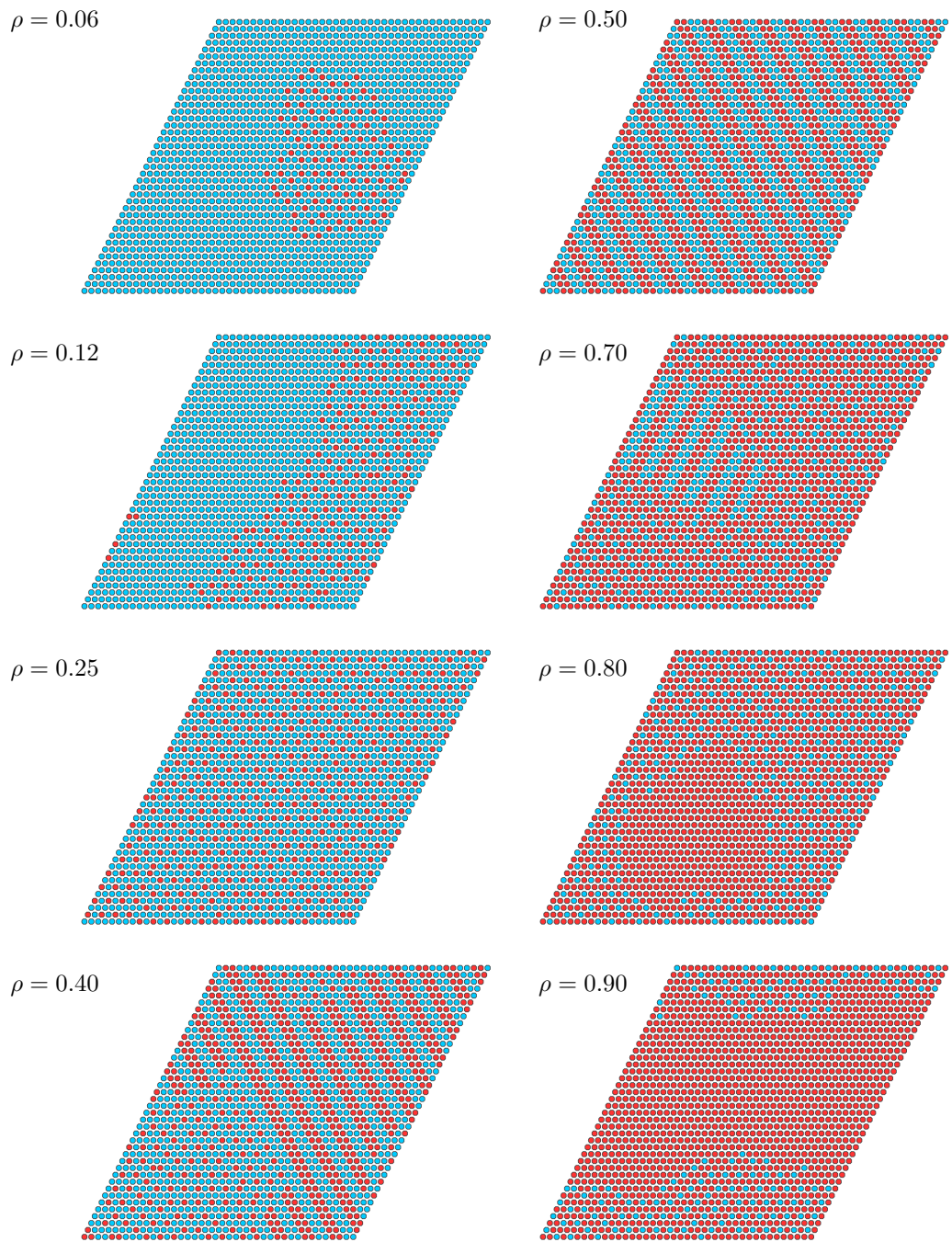


Figure 3: The emergent structures at various densities when we favor degree 4.

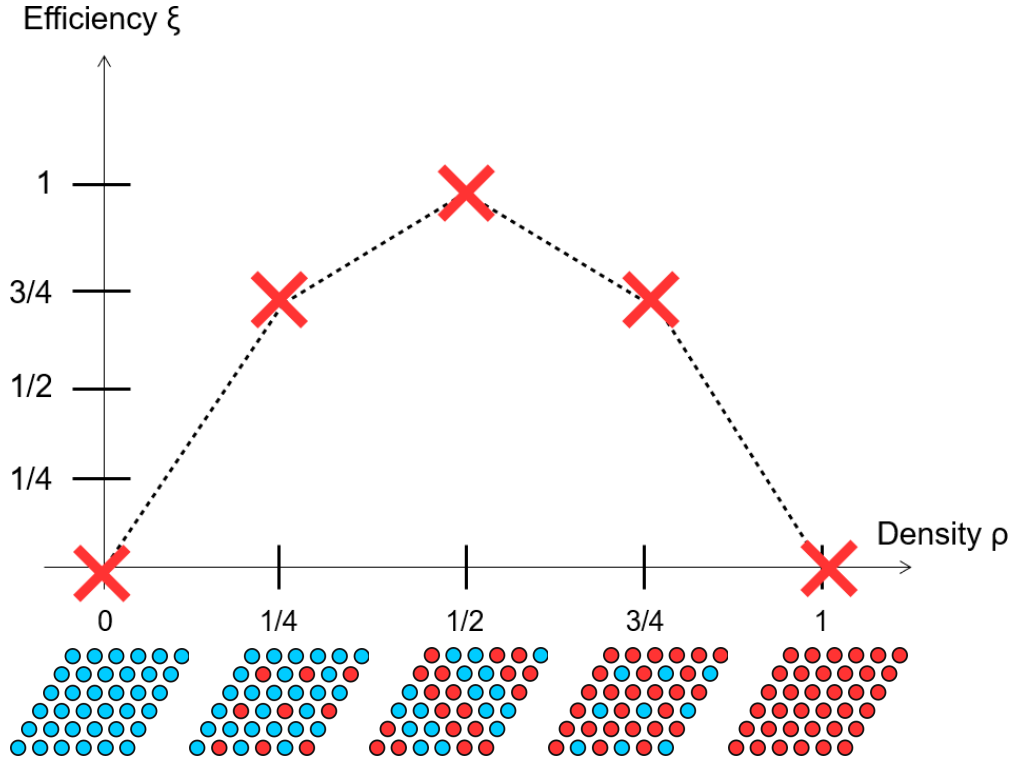


Figure 4: Plot of the obtainable efficiency ξ vs density ρ for homophily preference 4, with the five pure patterns.

specific density ρ , and a specific efficiency. We can plot these pure patterns on a graph of efficiency vs density. There are five such pure patterns for homophily preference 4, which we will first discuss since it represents the more common situation. These are marked as crosses on Figure 4.

Homophily preference 4: For densities in between pure patterns for homophily preference 4, such as $\rho = 0.45$ as shown in Figure 2.d, a mixture of two patterns is obtained. We expect the boundary between these patterns to have length on the order $O(\sqrt{N})$ between these patterns, where N is the number of sites. This means that the efficiency of these non-pure configurations, when averaged over the N sites, is asymptotically equivalent to the interpolated efficiencies of the two pure patterns it lies between. Thus, in between the pure patterns, we draw straight lines representing the optimal efficiency obtainable at each density, as the number of sites go to infinity. The efficiencies on the lines arise by mixing specific proportions of the two adjacent patterns.

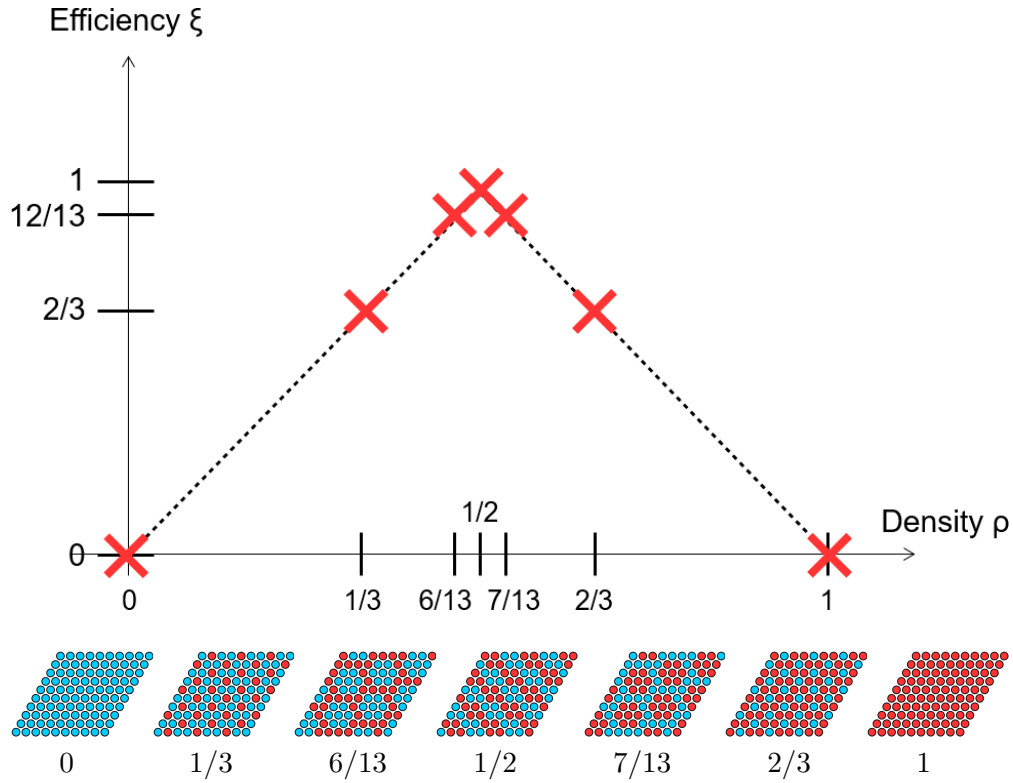


Figure 5: Plot of the obtainable efficiency ξ vs density ρ for homophily preference 3, along with the seven pure patterns corresponding to the red points.

Homophily preference 3: An unusual situation occurs in the case of homophily preference 3, however. The pure patterns correspond to densities 0, $1/3$, $6/13$, $1/2$, $7/13$, $2/3$ and 1, with efficiencies 0, $2/3$, $12/13$, 1, $12/13$, $2/3$ and 0, respectively. Plotting these on a graph of efficiency vs density, we find that the first four points and the last four points are collinear (see Figure 5). The significance of this is that on densities ρ that do not coincide with pure patterns, the interpolated efficiencies can be asymptotically achieved by a variety of mixtures of pure patterns. For example, at red particle density $\rho = 0.25$, one simulation may yield a mixture of the patterns corresponding to efficiencies 0 and $6/13$, while another may yield a mixture of the patterns corresponding to efficiencies $1/3$ and $1/2$. This is in contrast to the homophily preference 4 case, where patterns an intermediate density will always be a mixture of pure patterns immediately to the left and to the right on the plot. The mixed patterns arising from homophily value 3 give rise to far less predictable, but very intriguing, emergent behaviors.

References

- [1] S. Camazine, P. K. Visscher, J. Finley, and R. S. Vetter. House-hunting by honey bee swarms: Collective decisions and individual behaviors. *Insectes Sociaux*, 46(4):348–360, 1999.
- [2] S. Cannon, J. J. Daymude, C. Gokmen, D. Randall, and A. W. Richa. A Local Stochastic Algorithm for Separation in Heterogeneous Self-Organizing Particle Systems. *arXiv:1805.04599 [math-ph]*, June 2019. arXiv: 1805.04599.
- [3] S. Cannon, J. J. Daymude, D. Randall, and A. W. Richa. A Markov chain algorithm for compression in self-organizing particle systems. In *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing*, PODC '16, pages 279–288, Chicago, IL, USA, 2016. ACM.
- [4] N. J. Mlot, C. A. Tovey, and D. L. Hu. Fire ants self-assemble into waterproof rafts to survive floods. *Proceedings of the National Academy of Sciences*, 108(19):7669–7673, 2011.
- [5] T. H. Roulston, G. Buczkowski, and J. Silverman. Nestmate discrimination in ants: effect of bioassay on aggressive behavior. *Insectes Sociaux*, 50(2):151–159, 2003.
- [6] P. S. Stewart and M. J. Franklin. Physiological heterogeneity in biofilms. *Nature Reviews Microbiology*, 6:199–210, 2008.
- [7] G. Wei, C. Walsh, I. Cazan, and R. Marculescu. Molecular tweeting: Unveiling the social network behind heterogeneous bacteria populations. In *Proceedings of the 6th ACM Conference on Bioinformatics, Computational Biology and Health Informatics*, BCB '15, pages 366–375, New York, NY, USA, 2015. ACM.