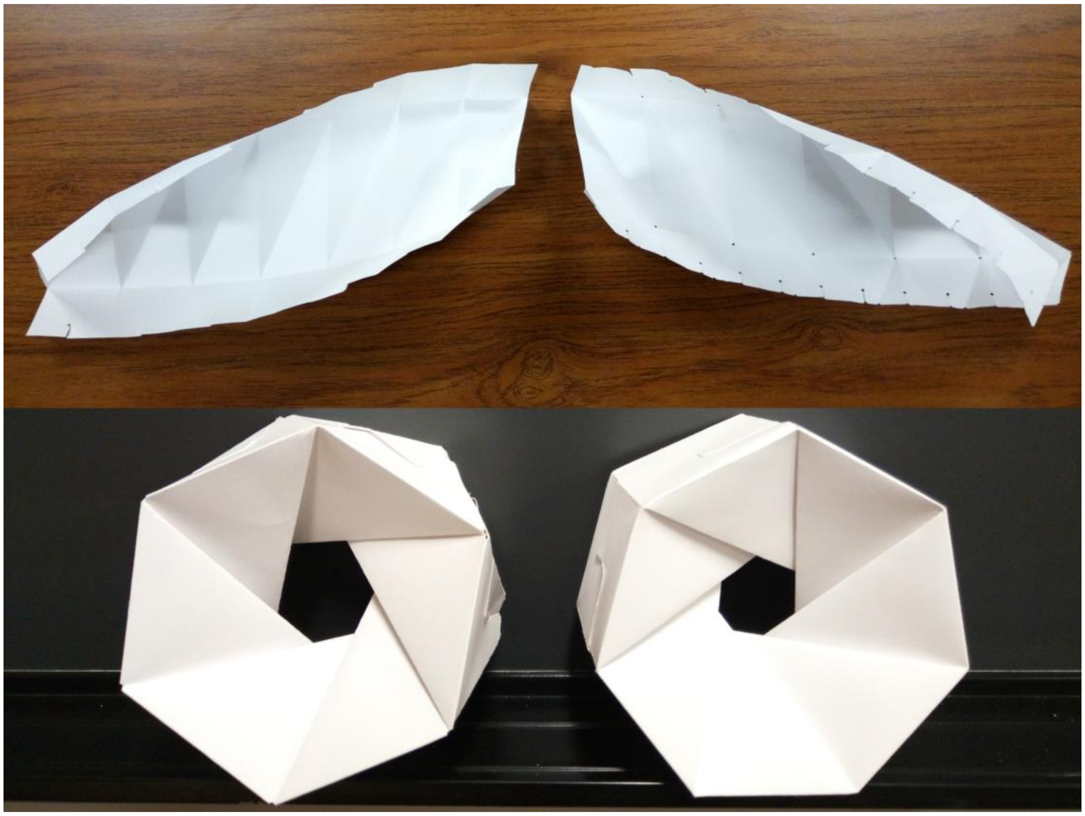

A flat torus is classically described as the geometric space obtained by gluing the parallel sides of a square, with the same orientation. One can start instead from any parallelogram, so that there is a whole family of flat tori. In mathematical terms, one can think of modding out the plane by the group generated by two translations.

The family of flat tori obtained this way has itself a rich geometric structure. It is an orbifold and it is called the modular curve (of tori). It is smooth almost everywhere except for two cone points corresponding to the square torus and the torus glued out from the 60-degree-angled parallelogram.

The precise mathematical construction of the gluing of the flat tori poses no particular issue. However, it is trickier to realize an isometric embedding of such a flat torus in euclidean 3-space. Recent work of Borrelli et al., using Nash embedding theorem, show how to realise a C1 embedding, but this embedding has a smooth fractal structure which is difficult to realize in practice, especially in paper!

We are looking here for a different type of embedding, an origami embedding, i.e., a continuous, piecewise linear embedding of the flat torus into the euclidean 3-space. The very existence of a non-trivial embedding of this kind is not obvious at all (in fact, till 3 months ago, we thought that such a locally flat embedding did not exist).

In late 2019 at the "Illustrating Mathematics" semester in ICERM, we learned about two possible realizations of flat torus embeddings with paper folding.

One realization was explained to us by Henry Segerman who learnt about it from a dedicated page on the French website mathcurve.com, with origins that can be traced to a Russian paper by Burago and Zalgaller from 1996, translated into English in 1997. The method consists in starting with a regular polygonal prism, take only its external walls, and then connect top and bottom polygons with congruent triangles. We will call these embeddings, prism embeddings; and the embedded tori, prism tori.

Glen Whitney showed us that a similar construction is doable starting from an antiprism, giving rise to antiprism tori.

In the gift exchange, we propose to give away flat layouts for flat tori. Our layouts have been thoroughly tested. They contain some excess paper and flaps so that once folded, they stand stably on their own, even without glue.

We also want to contribute a short paper about flat tori to the book. This will include a short history of the paper flat tori concept, then explain the buildup of the layout and the set of points in the modular curve covered by our paper flat tori. The set of flat tori that can be realized as polyhedral flat tori is still not completely clear.

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References:

- "Tore plat", article on the MathCurve.com website.

http://mathcurve.com/polyedres/toreplat/toreplat.shtml

- Joseph O'Rourke.

On Folding a Polygon to a Polyhedron.

2010.

https://arxiv.org/abs/1007.3181

- Burago, Zalgaller.

Isometric piecewise linear immersion of two-dimensional

manifolds with polyhedral metrics into RR^3.

St Petersburg Math J., vol 7, no 3, 369--385, 1996.

- Henry Segerman. Visualising mathematics with 3D printing. pp 127 sqq.

- Vincent Borrelli, Saïd Jabrane, Francis Lazarus, and Boris Thibert.

Flat tori in three-dimensional space and convex integration.

Proceedings of the National Academy of Sciences, 2012.

https://doi.org/10.1073/pnas.1118478109