Topological Dance Puzzles
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Solutions to most problems are found in my paper "Dancing Topologically" in the Bridges Math and Art Conference Archives, 2021. Bridges 2021 Proceedings. Pp 79-86. http://archive.bridgesmathart.org/2021/bridges2021-79.html. Some problems ask you to create a dance phrase to match a knot or link; that can be as much or more fun than the mathematical part of the puzzle!

1. Imagine that a dancer traversing a path on the stage trails behind a string, Fig 1. The dancer continues to move and trail the string until reaching the dancer's starting point P. The dancer is careful never to move directly over a crossing point of previously trailed strings such as point C by moving slightly to one or the other side of the crossing instead, as indicated by one of the solid arrows. In the figure the dancer then proceeds to starting point P by following one of the dotted arrows and connects the two ends of the string to create a loop. Is it possible that a loop created in this way by a dancer dragging a string might in some cases be a knot (that is NOT the unknot)!? Notice that in the figure the fact that earlier strands of the string pass under later strands is indicated by the "break" in the underlying string segment. It turns out that the decision as to which bold arrow the path takes makes no difference as to whether the loop created in Fig. 1 is knotted, or which knot it might form.

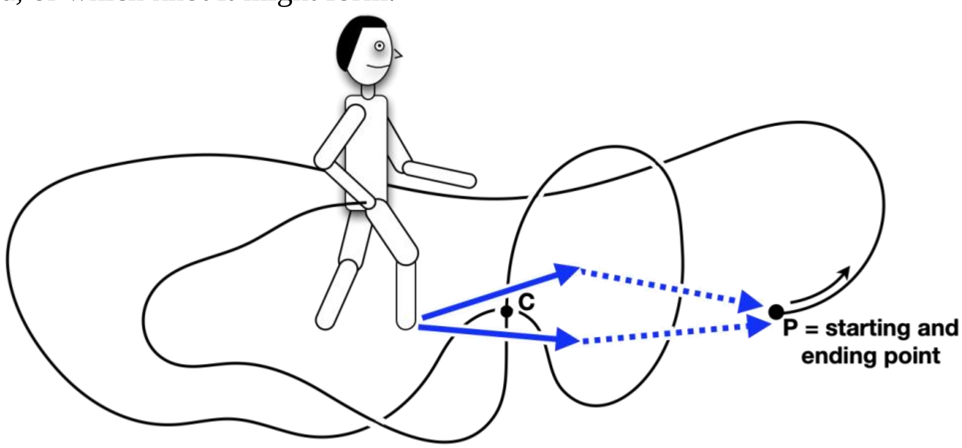


Figure 1

2. Imagine now that each of a pair of dancers trails a string behind themself as they move through a complex floor pattern, never move over a point where two strings already cross, and finally come to rest at either their own or their partner's starting point. Then the string paths will combine to take the form of a knot or link. We say that the diagram produced represents a knot or link that is *duet*- or *2-danceable*. We might turn the exercise around and ask, "Which knot or 2-link diagrams are so danceable by two dancers?"

   (A) Explain why every two-crossing knot is solo-danceable - are they all the unknot? Show that every two crossing 2-component link is 2-danceable.

   (B) Find a three-crossing unknot that is not solo-danceable but is 2-danceable. Perform it with a partner. You should have found in problem 1 that solo-danceable knots are always the unknot, but this problem asks you to find a diagram of an unknot that is not solo-danceable. Generally, knot theorists look for properties of knots that hold for all diagrams of a particular knot, but this shows that danceability is NOT one of those properties!

(C) Work in a group of four. Find a highly symmetrical way to dance Solomon's knot, Fig. 2, so that two dancers circle clockwise on one link and two circle counterclockwise on the other link. This pattern is called the "Hey for four" in contra dance.

Figure 2

(D) Work in a group of three or four. Find a 3-danceable pattern for the 3 by 3 Celtic knot (Appendix A) and develop, rehearse, and perform it. It is sometimes helpful to have one person stand out and help direct, which is why a group of four might be helpful.

3. Examine the two diagrams in Fig. 3, which represent the 8-crossing knot that is given the standard label 8_{18} in most knot catalogues. 8_{18} is said to be an 8-crossing knot because 8 is the minimal number of crossings in any of its planar diagrams; however, an *n*-crossing knot may have *n*-crossing diagrams that look very different! Every knot has a *braid* diagram like that for 8_{18} shown in Fig. 4. We imagine that the two strands labeled P are connected, as are the two labeled Q and the two labeled R (without adding additional crossings).

   (A) Explain how 8_{18}'s braid diagram is generated from its standard diagram on the left (hint: pay attention to the three small line segments crossing the knot.

   (B) Find the value *m* such that the 8_{18} diagram is minimally *m*-danceable: every braid with *m* strands is *m*-danceable (can you see why?), but is the 8_{18} diagram also 2-danceable? Explain your reasoning. Remember to pay attention to the direction in which the dancers travel.

   (C) The braid form of 8_{18} also represents the "three-person weave" pattern used in basketball. Watch the video clip https://www.youtube.com/watch?v=DEyULvNXBmo at "Warriors Weave," 2015, in which the Golden State Warriors NBA basketball team uses the three-person weave and explain its relationship to the braid diagram for 8_{18}.

   (D) In a group of three make up a dance phrase that uses the three-person weave.

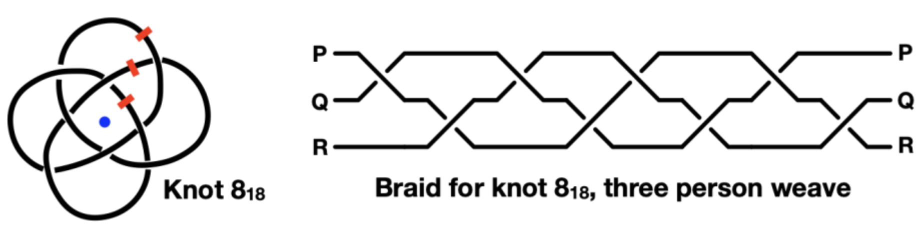
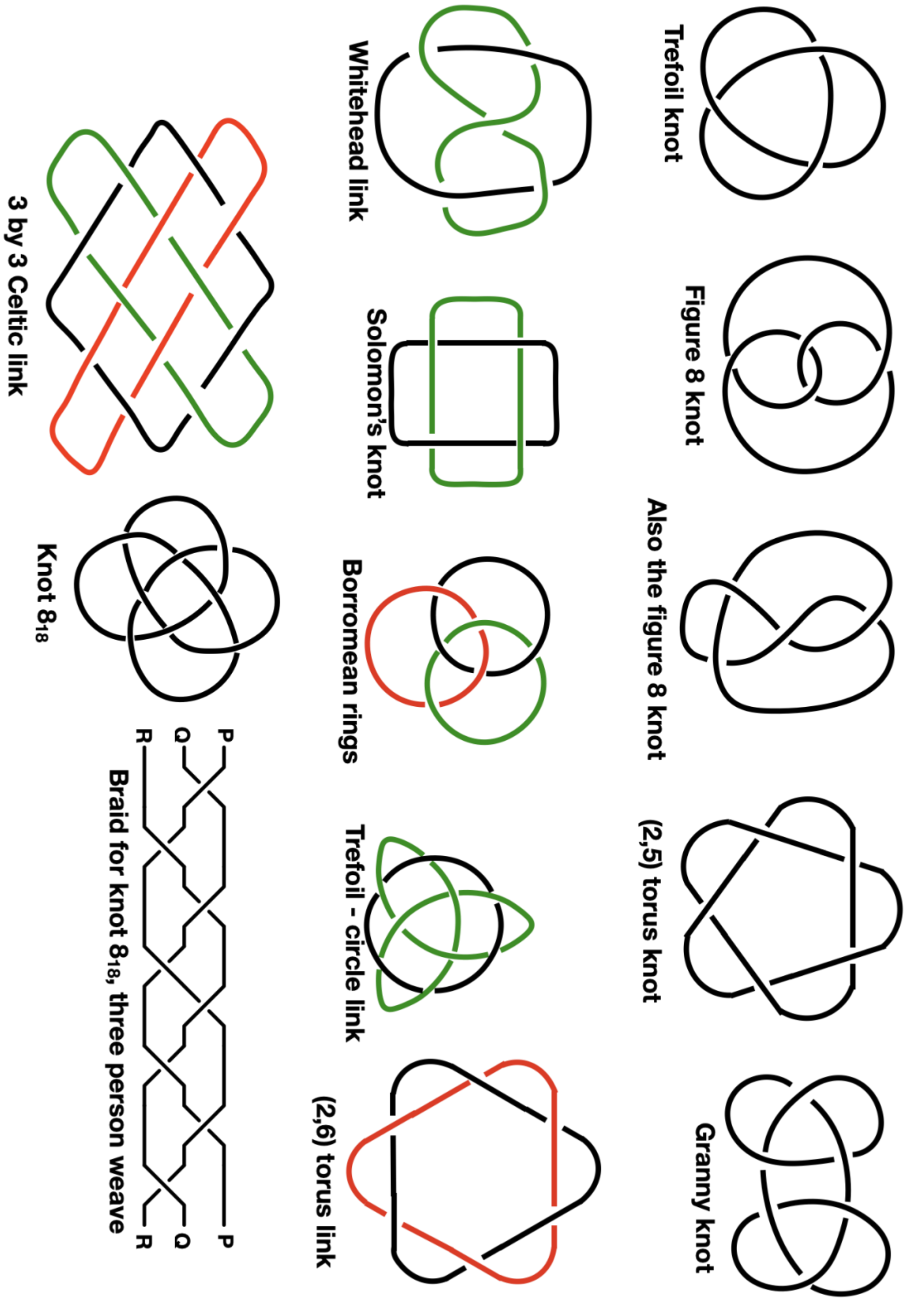


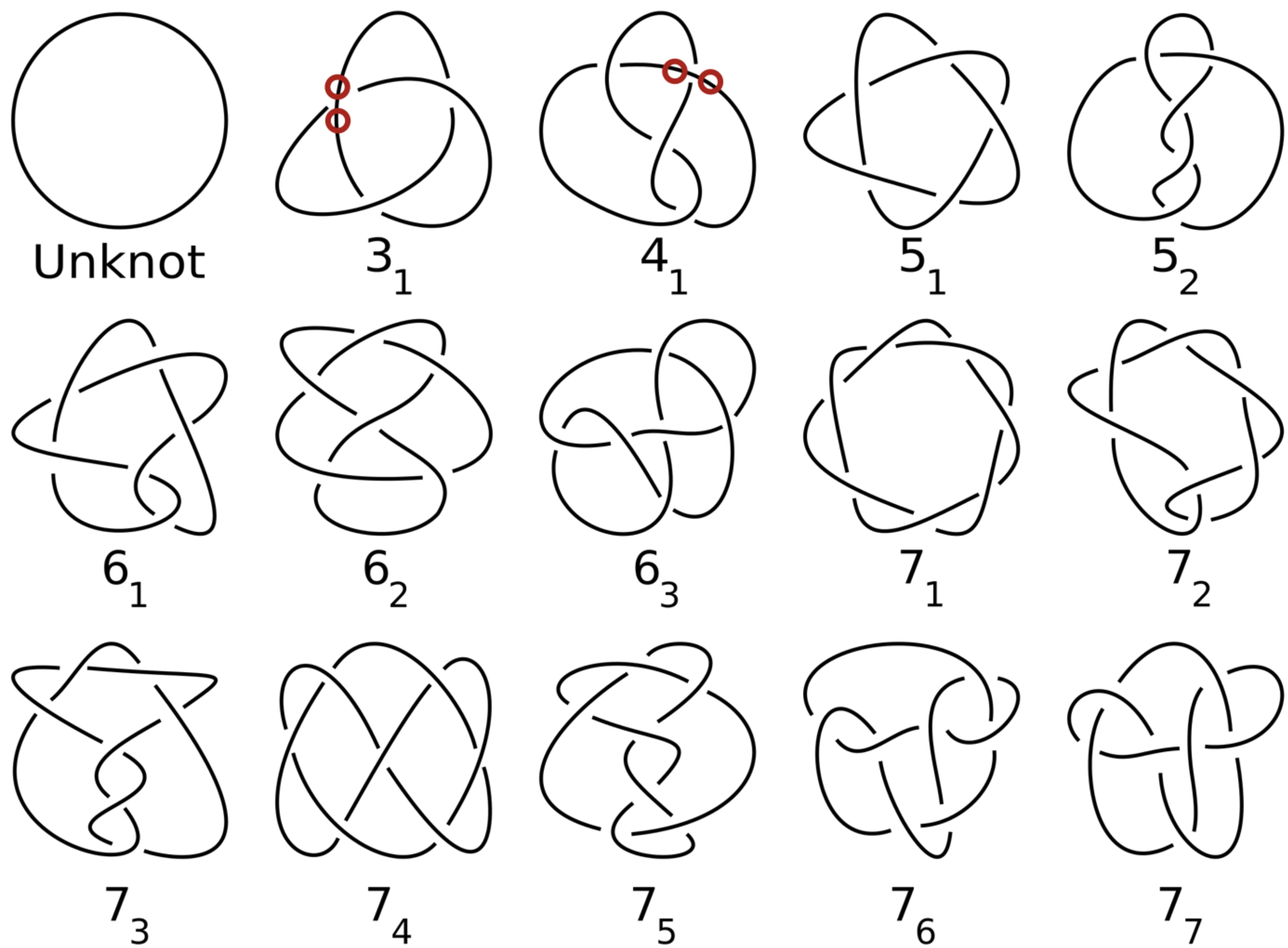
Figure 3: Knot 8_{18} and its braid form, similar to the three-person weave.

4. For each knot or link drawing in Figures 4 and 5 that you have not yet considered find the minimal *n* such that the drawing is *n*-danceable in each direction. Choose one or two as the basis for a dance phrase. Figure 5 shows diagrams for all prime knots with minimal crossing number less than seven; find "string duet-danceability" as defined in the caption for each.

Figure 4. Knot and Link Examples



Use a pair of red circles to show “string duet-danceability,” if possible. A knot diagram is string duet-danceable if two dancers can start at a crossing point and move away from that point in opposite directions along the same strand, obeying under and overcrossing rules, until they traverse the entire diagram and meet again. Find which knots are so danceable. Also for each knot not yet considered and each direction on its diagram find *n* and starting points such that the knot diagram is *n*-danceable. Some diagrams may be minimally *m*-danceable in one direction and minimally *n*-danceable in the other direction with *m* and *n* unequal. String duet danceability is shown for 3_1 and 4_1, the trefoil and the figure 8 knot.



Alexander and Briggs knot notation

Figure 5