A Way to Derive the Spidron Formulas

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Abstract: 20 years ago Dániel Erdély asked me to visualize the behavior of his invention, the (hexagonal) Spidron[1], a shape that tiles the plane, and is composed of an infinite number of increasingly small triangles. The Spidron System can be twisted up to form a periodic 3D landscape, without distorting its triangles. To visualize this, I needed to derive the equations, or formulas, that describe the transformation. This paper presents the derivation of three of the most important formulas, according to the way I followed. Later others (L Szilassi and M Hujter) used different approaches and achieved equivalent versions of the main formula presented here.

1. About the hexagonal Spidron System

A single *Spidron* is an S shaped form composed of an infinite number of increasingly small triangles (see **Figure 1**, which has three adjacent S-shapes in the middle). The two end points of the S shape are the points of convergence for the ever-shrinking triangles.

If we cut the S-shape in two, we get the so-called *Semi-Spidron* (shaded in **Figure 2**), which has only one point of convergence. There are two types of triangles in it: isosceles and equilateral ones; they follow each other in an alternating way as they converge. We can place six of these Semi-Spidrons next to each other, to get a regular hexagon. They share their points of convergence at the hexagon's center. Since the hexagon tiles the plane, the Semi-Spidron and the S-shape tile it too, creating a *Spidron System*. **Figure 1** shows three adjacent hexagons from the system.

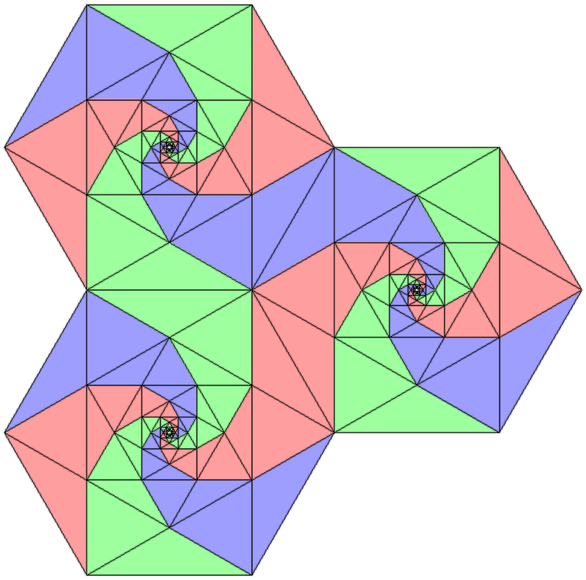


Figure 1

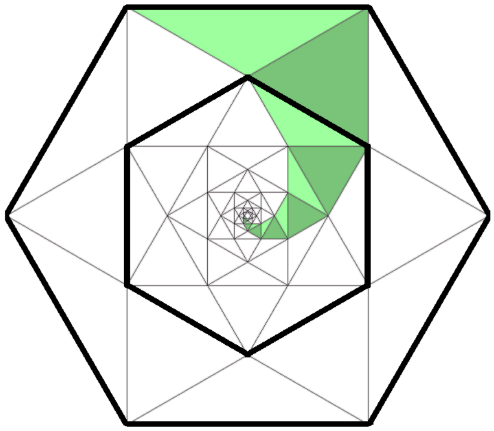


Figure 2

A whole hexagon is called a *Spidron-nest*. In the context of a Spidron-nest, a Semi-Spidron is also called a *Spidron-arm*. The cluster of triangles that form a circle (highlighted with thick black lines in **Figure 2**) is called a *Spidron-ring* (or belt).

1 Spidron™ is currently (June, 2018) a registered trade mark.

1.2 Flexibility

The hexagonal Spidron System has the ability to transform so that the shape of its triangles doesn't change. During this transformation it twists up into a periodic 3D landscape, or relief (**Figure 3**). This paper analyzes this phenomenon, by looking for the right formula, or formulas, that describe the transformation.

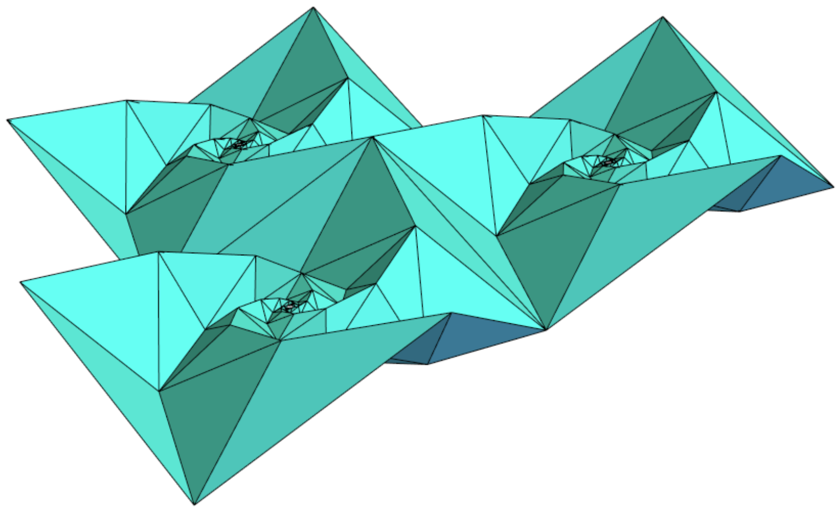


Figure 3

For further introduction to the Spidron see [1] and [2].

2. The First Formula

Let's start with the planar state of the system. Select three, mutually adjacent Spidron-nests, and concentrate on the junction point where 3 diamonds meet (highlighted on **Figure 4**).

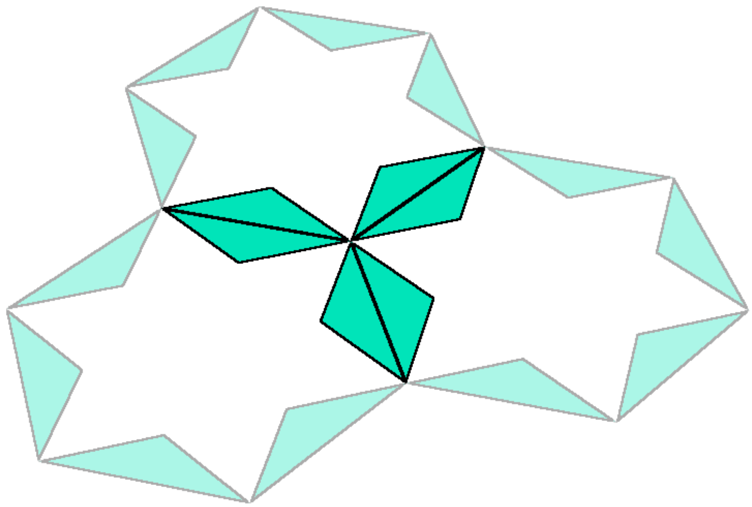


Figure 4

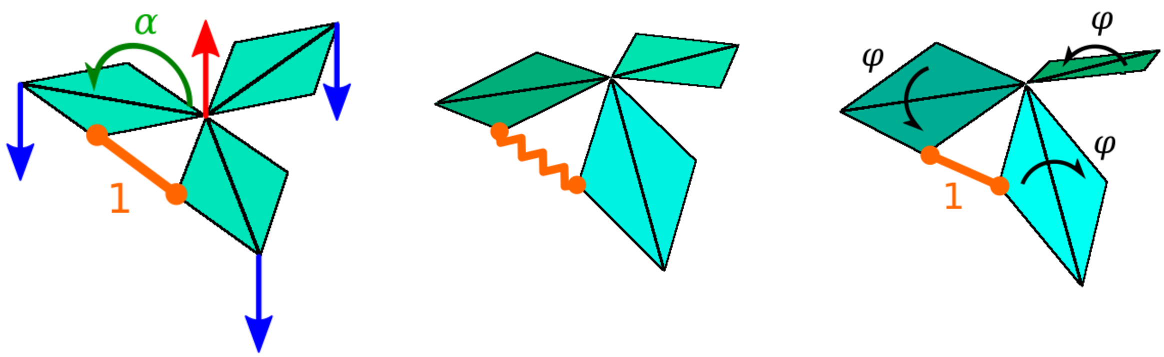
Each diamond is a pair of adjacent isosceles triangles that belong to adjacent Spidron-nests. We assume that the adjacent triangles in each pair move together, continuing to form rigid diamonds. This is a valid assumption, leading to a valid realization of the Spidron-transformation.

2.2 Leaving the plane

Let's assume that the junction point is going upwards, and the diamonds' far ends are going downwards. In terms of rotations, each diamond is getting inclined with the same α angle around axes parallel to their shorter diagonals (see **Figure 5a**).

α corresponds to the inclination angle of the hexagon's edges at the perimeter of the Spidron-nest (the lower-left one from **Figure 4**, but this applies to others in the system too[2]), relative to original plane. This is our *independent variable*. We would like to choose it freely, at least within reasonable limits.

Remember that the marked segment must retain its (say, unit) length, as there are triangles on both of its sides, which we only hid for the illustration.



a *b* *c*

Figure 5

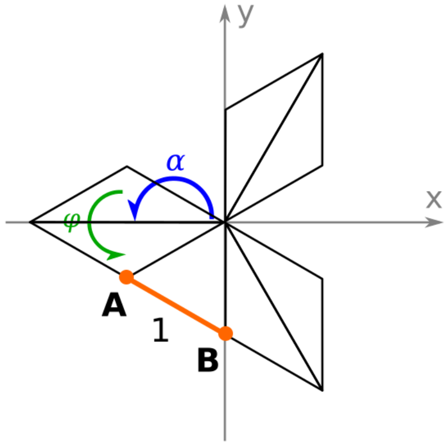
If nothing else happens, the segment cannot retain its length, it breaks (**Figure 5b**). But if we allow the diamonds to turn around their (now inclined) longer axes, with the same φ angle for each one (where the direction of the 3 rotations is symmetric, or coherent, in a circular fashion), the length of the segment can be maintained (**Figure 5c**).

This φ can, and must, be chosen well. If we manage to find a suitable value (or formula, which depends on α), the shape of the triangles in the outermost Spidron-ring will be preserved. The question is: what is this φ?

2 Sometimes I speak of a single Spidron-nest only, due the fact that the same thing happens to every Spidron-nest in the system.

2.3 Formalizing

Let’s label one end of the above segment with A, and the other with B (**Figure 6**). Their coordinates are shown. In the figure, the X and Y axes lay in the plane of the image, as indicated, and the Z axis is perpendicular to that, pointing outwards.



$$\underline{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Figure 6

We state that no matter what happens to the original $\underline{A}$ and $\underline{B}$ points, the distance of their transformed counterparts ($\underline{A}$’ and $\underline{B}$’) must always be 1:

$$\left| \underline{A}' - \underline{B}' \right| = 1 \qquad (1)$$

This is our main equation.

We are going to apply various rotations to the vectors around the X, Y and Z axes, by multiplying them with these known rotation matrices from the left:

$$\underline{\underline{\mathrm{Rot}_{\mathrm{X}}}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\underline{\underline{\mathrm{Rot}_{\mathrm{Y}}}}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let’s express the transformed $\underline{A}$’ (see **Figure 6**). The original $\underline{A}$ vector is first tilted by φ around the X axis, and then rotated by –α around the Y axis (α is considered to be a positive value). In the formula, these rotations follow each from right to left:

$$\underline{A}' = \underline{\underline{\mathrm{Rot}_{\mathrm{Y}}}}(-\alpha)\ \underline{\underline{\mathrm{Rot}_{\mathrm{X}}}}(\phi)\ \underline{A} \qquad (2)$$

The transformed $\underline{B}'$ is expressed very similarly, but we temporarily align its diamond with the Y axis first, by applying a rotation around Z (see **Figure 7**). This way we won't need to perform rotations around oblique axes.

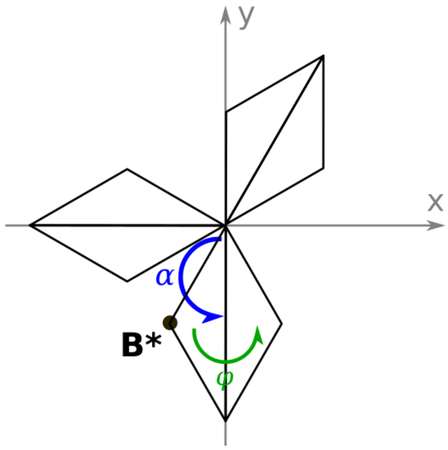


Figure 7

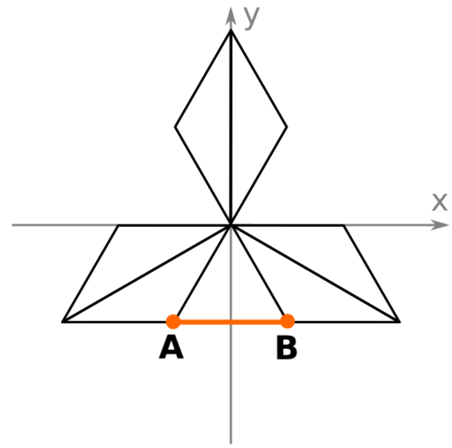


Figure 8

This correction and back-correction is valid, and doesn't interfere with the result. The Z-heading of the diamonds doesn't change when the Spidron transforms. In other words, if we project the diamonds' longer diagonals on the original plane (the plane of the image), the direction of the projections don't change during the transformation (they only get shorter). And after the previous back-correction, we too get back the original Z-heading of $\underline{B}$'s diamond.

So we first apply the aligning rotation (-30° around Z), then we tilt the aligned $\underline{B}$* by φ around the Y axis, and then rotate it with α around the X axis. Finally, we compensate the previous alignment applying a rotation by 30° around Z:

$$\underline{B}' = \underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(30^\circ)\ \underline{\underline{\mathrm{Rot}_{\mathrm{X}}}}(\alpha)\ \underline{\underline{\mathrm{Rot}_{\mathrm{Y}}}}(\phi)\ \underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(-30^\circ)\ \underline{B} \qquad (3)$$

But before going on, let's apply one more alignment, this time a permanent one. Let's rotate the whole system with 30° around Z, thus aligning $\overline{\mathrm{AB}}$ with the X axis (**Figure 8**). This will make one of our immediate results (6) simpler (and more useful too). Because of this alignment, $\underline{A}'$ gets an additional rotation, and the back-correcting rotation of $\underline{B}'$ changes from 30° to 60°:

$$\underline{A}' = \underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(30^\circ)\ \underline{\underline{\mathrm{Rot}_{\mathrm{Y}}}}(-\alpha)\ \underline{\underline{\mathrm{Rot}_{\mathrm{X}}}}(\phi)\ \underline{A} \qquad (4)$$

$$\underline{B}' = \underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(60^\circ)\ \underline{\underline{\mathrm{Rot}_{\mathrm{X}}}}(\alpha)\ \underline{\underline{\mathrm{Rot}_{\mathrm{Y}}}}(\phi)\ \underline{\underline{\mathrm{Rot}_{\mathrm{Z}}}}(-30^\circ)\ \underline{B} \qquad (5)$$

These are the final formulas for $\underline{A}'$ and $\underline{B}'$.

Below is the $\overline{\text{A'B'}}$ vector (or, to be more precise, $\overline{\text{B'A'}}$ vector):

$$\underline{A}' - \underline{B}' = \begin{pmatrix} \frac{1}{2}\left(-3\cos(\alpha) + \cos(\phi)\right) \\ \frac{1}{2}\sin(\alpha)\sin(\phi) \\ -\cos(\alpha)\sin(\phi) \end{pmatrix} \qquad (6)$$

And the following is the $\overline{\text{A'B'}}$ vector's length. We want to make sure that it is 1, complying with our main equation (1):

$$\left|\underline{A}' - \underline{B}'\right| = \frac{1}{2}\sqrt{\left[-3\cos(\alpha) + \cos^2(\phi)\right]^2 + \left[4\cos^2(\alpha) + \sin^2(\alpha)\right]\sin^2(\phi)} = 1$$

If we solve this equation, φ can be expressed as a function of α like this (there are more than one solutions, but they seem to be equivalent[3] apart from differences in sign, which mean that the direction of the tilting can be reversed):

$$\phi = \arccos\left(2 - \frac{1}{\cos(\alpha)}\right) \qquad (7)$$

To achieve this formula, 20 years ago (in 1998) I used a program called *Derive*, and it needed some help from me. This year I used the *Wolfram Language*, and it solved the equation on its own.

Using this formula for φ ensures that the $\overline{\text{AB}}$ segment keeps its length, and the triangles *in the outermost Spidron-ring* don't get distorted, if we set our independent α variable to any arbitrary, reasonable value. This is the first formula.

2.4 Generalizing the angles to inner rings

You may ask: What about other, inner Spidron-rings? The above formula works for any Spidron-ring in the Spidron-nest, if we interpret it properly (we need somewhat new definitions now, which don’t involve diamonds[4]): it gives the rotation angle of the ring’s isosceles triangles around their longest edges (i.e. their bases), in the function of the inclination angle of these base edges relative to the original plane. These base edges are the hexagon’s edges at the outer perimeter of the Spidron-ring we are currently examining.

3 My main goal was to get a working visualization, and, apart from a few numerical tests, I didn’t perform thorough mathematical analysis.

4 In fact we could have operated with isosceles triangles (instead of diamonds) in the first place, concentrating on one single Spidron-nest only, but I found the idea of the junction point more intuitive because it captures more of the symmetry.

3. The Second Formula

At the end of the previous section we generalized α and φ. Let's focus more on α and its generalization. Our independent variable is called α. This is the inclination angle of the hexagon's edges at the outer perimeter of the outermost Spidron-ring (the Spidron-nest itself). There is an angle analogous to this at the outer perimeter of the *second*-outermost Spidron-ring[5]. We may call it α_2 (see **Figure 9**). What we are looking for now, is a formula that gives α_2 in the function α.

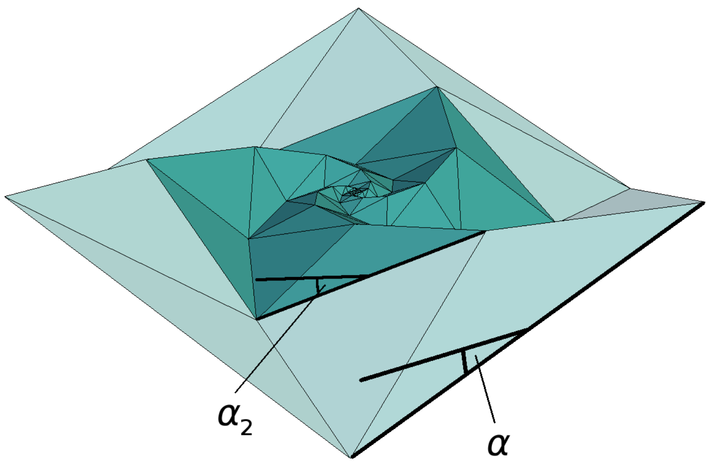


Figure 9

Since the Z coordinate of the $\overline{A'B'}$ vector (in equation (6)) represents the height-difference that B' gains over A', if we take the *arcus sinus* of this Z coordinate, we get the inclination angle of the unit-length $\overline{A'B'}$ vector relative to the XY plane, which is exactly what we are looking for. After substituting (7) in the place of φ, we get this for α_2:

$$\alpha_2 = \arcsin\left(\sqrt{-1 + 4\cos(\alpha) - 3\cos^2(\alpha)}\right) \qquad (8)$$

In fact, this formula is valid for any consecutive pair of α_{n+1} and α_n, so it provides a recursive solution to the series of α_n's. In other words, it gives the connection between the inclination angle of the hexagon's edges at the *outer* perimeter of a Spidron-ring, and the similarly defined angle at the *inner* perimeter of that ring. This is the second formula in my paper, and is generally considered to be the main Spidron Formula.

In 2004, Prof. Lajos Szilassi presented an independent derivation [3], which led to a formula equivalent to (8).

In June 2018, Mihály Hujter, PhD, described a different derivation [4], again leading to a formula which is equivalent to (8). Its English version is yet to be published at the time of writing this manuscript.

5 Which happens to be the inner perimeter of the outermost ring.

4. The Third Formula

Another important angle is the amount the second-outermost Spidron-ring turns around Z, relative to the Spidron-nest it belongs to, whose orientation we consider to be fixed.

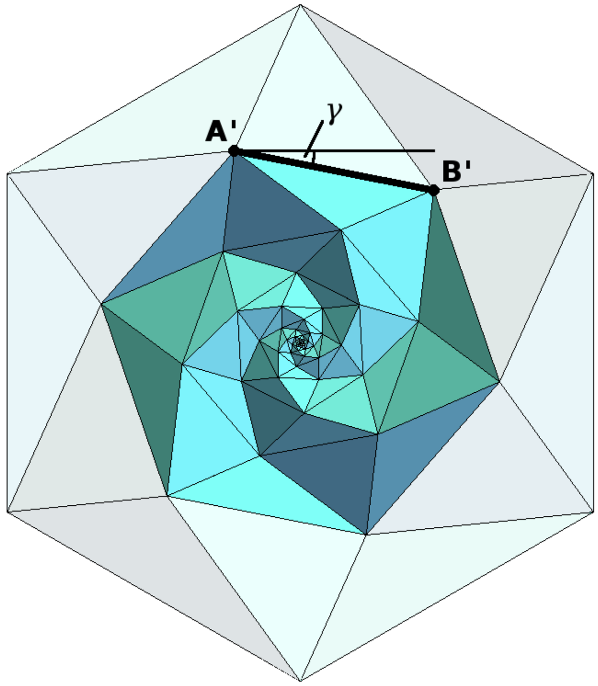


Figure 10

This is where aligning the original $\overline{\mathrm{AB}}$ segment to the X axis becomes useful (**Figure 10**). It is now easy to read this angle (which I named *γ*) from the coordinates of the $\overline{\mathrm{A'B'}}$ vector (6). We just have to take the *arcus tangent* of the ratio of its Y and X coordinates, and get:

$$\gamma = \arctan\left(\frac{\sin(\alpha)\sin(\phi)}{-3\cos(\alpha)+\cos(\phi)}\right)$$

This angle too can be generalized to inner Spidron-rings.

References

[1] Erdély, Dániel: Concept of Spidron System, *Proceedings of "Sprout-Selecting"Conference: Computer Algebra Systems and Dynamic Geometry Systems in Mathematics Teaching*, Pécs, Hungary, 2004, pp. 68–77.

[2] Erdély, Dániel: Some Surprising New Properties of the Spidrons, *Bridges Conference Proceedings,* 2005, pp. 179–186.

[3] Szilassi, Lajos: The right for doubting – and the necessity of doubt, *Proceedings of "Sprout-Selecting"Conference: Computer Algebra Systems and Dynamic Geometry Systems in Mathematics Teaching,* Pécs, Hungary, 2004, pp. 78–96.

[4] Hujter Mihály: A csillaghatszög spidronszerű felgyűrődése mögötti matematika, *"Haladvány Kiadvány"*, June 16, 2018 (in Hungarian) http://math.bme.hu/~hujter/180616.pdf