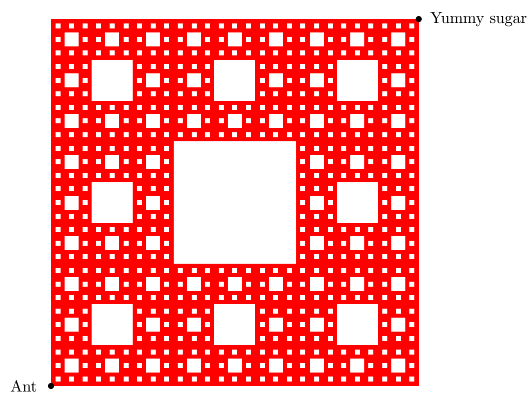
Traveling through the Sierpinski carpet and Menger sponge

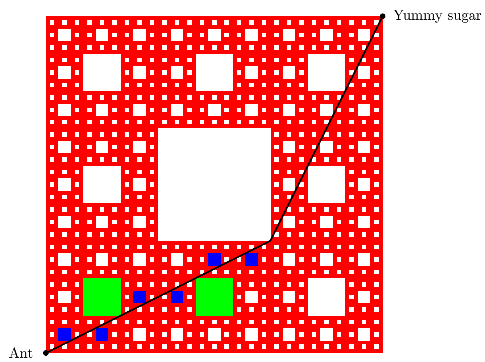
Derek Smith, Lafayette College
smithder@lafayette.edu

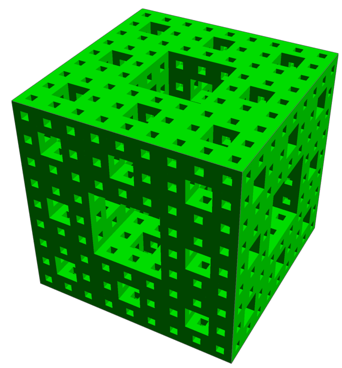
The Sierpinski carpet is an intriguing 2-dimensional fractal. You can construct the carpet by taking a solid square of side length 1, dividing it into nine sub-squares of side length 1/3, and removing the "open" sub-square in the middle, *i.e.* remove all of the points of that middle sub-square except for the points on its boundary edges. For each of the eight remaining sub-squares, repeat the procedure above by dividing it into 9 sub-sub-squares and removing the middle one; and for each of the 64 remaining sub-sub-subsquares repeat the procedure; and so on; keep going; you're not done yet! The limiting object is the Sierpinski carpet. The red figure below is just an approximation of it, after repeating the subdividing and removal procedure only 4 times, but it should be enough to give you a good sense of things.



Problem 325 in the April 2015 issue of *Math Horizons* asked the following question. An ant starts at one corner of the fractal and wishes to travel to the opposite corner while staying on the fractal. It's clear that the ant can do this by traveling a distance of 2, simply by moving along two exterior edges. But can the ant get to the opposite corner by a shorter path? What is the shortest path the ant can take to get from one corner of the Sierpinski carpet to the opposite corner while staying on the fractal and not falling into any open square hole? **Don't go to page 2 until you've tried your best to get a path whose length is shorter than 2!**

In fact, the ant can do much better than a distance of 2 by going through the interior of the carpet. If the only paths through the carpet use line segments parallel to the outer edges of the carpet, the ant wouldn't be able to beat a distance of 2. But there are line segments in the carpet that have slopes of 1/2, 1, 2, and their negatives! The figure on the left below shows one of two shortest possible paths from the lower left corner to the upper right corner, with a total distance of $(2/3)\sqrt{5} \approx 1.49$. Try to use the self-similar structure of the carpet and the positions of the green and blue holes relative to the lower line segment of slope 1/2 to convince yourself that this path is, indeed, contained in the Sierpinski carpet. Also, try to find a line segment in the carpet of slope 1!





A 3-dimensional fractal that is closely related to the Sierpinski carpet is the Menger sponge: it is a cube with Sierpinski carpets on its faces, and with open tunnels bored straight through the cube where there are open square holes in the carpet faces. An approximation of the Menger sponge is shown above in green.

The problem in *Math Horizons* had a second part. Suppose a termite wants to travel from one corner of the Menger sponge to the opposite corner. A path of length 3 can be had by following three exterior edges of the sponge... but can the termite do better? Maybe it's via a path that stays on the outer Sierpinski carpet faces of the sponge, but we also allow the termine to bore through the sponge, always staying in the green material of the sponge and avoiding any removed open tunnels. What's the shortest possible route?

Try to use what you now know about paths in the Sierpinski carpet to help you find a path that stays on the surface of the Menger sponge and has length less than 5/2. Then, for the real challenge: try to find a path that can be bored through the sponge and has length less than 2!

Good luck! If you would like some hints, or some good references on geodesic paths in fractals, just send me an email. As for the shortest possible path in the Menger sponge from one corner to the opposite corner... this problem remains unsolved.

This work is joint with Ethan Berkove.