Development of the Loyd Polyominoes Puzzle

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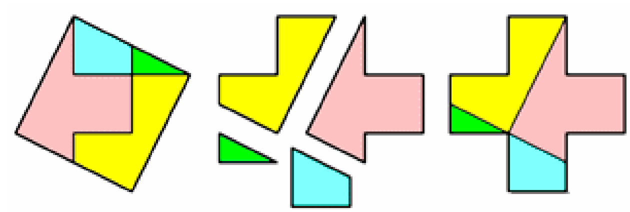
Summary

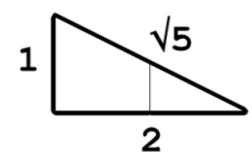
There are 5 tetrominoes and 12 pentominoes—17 polyomino shapes in total. The challenge is to find a group of only eight puzzle pieces that can make each of these 17 shapes. This design task is actually much more difficult than the ***Loyd Polyominoes Puzzle*** itself. There is a companion web site with downloadable files and other material at: **http://www.marchland.org/loyd**

Background

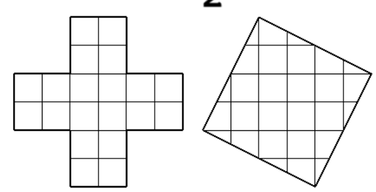
About 100 years ago, Sam Loyd showed how to dissect a Greek Cross to a square in only four pieces. Note the small green triangle.



This is the starting point for quite a complex project, so it is necessary to give precise definitions to all the words being used, particularly those referring to assemblies of things. At one point we will have to consider collections of collections of collections. To be precise, seventeen "sets" of "groups" of "pieces". Each technical term will be highlighted in **CAPITALS AND BOLD** on first use. The word **SET** will be used in its mathematical sense of a collection of objects, no two of which are identical. But other technical words will just be defined as they are used. As Humpty Dumpty said in ***Alice in Wonderland***, "When I use a word, it means just what I choose it to mean."



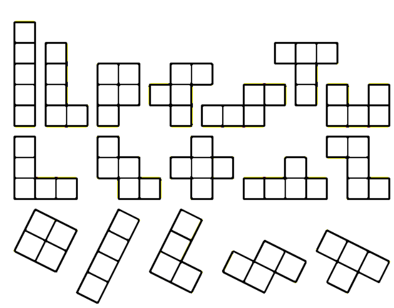
If the short side of the small green triangle is one unit, then its sides are 1, 2 and √5. It is one of two **BUILDING BLOCKS**. The other one is the unit square. Both of them have an area of 1 unit.



So, if the Greek Cross is 6 units wide and 6 units high as shown, then its total area is 20 units. This means that each side of the big square is √20 or 2√5 and its perimeter is 8√5.

For this project, a puzzle **PIECE** is made from one or more building blocks. There are well over 30 plausible shapes for puzzle pieces made in this way. The useful pieces have an area of 1 to 4 units. Several puzzle pieces can be put together to make a target **SHAPE**, like Loyd's Greek Cross or Square. The collection of pieces will be called a **GROUP.** The purpose of this project will be to identify a group of pieces that can make many target shapes. So, in the case of the Loyd Greek Cross and Square, the **group** of four **pieces** can make both of the two target **shapes**.

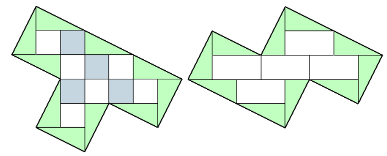
The Greek Cross is one of the **PENTOMINOES**, shapes made using five squares. There are 12 of them, usually known as I, L, P, R, S, T, U, V, W, X, Y and Z. Some people use the letter "F" for the R pentomino, and "N" for the S pentomino. The square is one of the **TETROMINOES** and there are 5 of them—square, I, L, skew and T. They are shown sloping to the right to match the Loyd dissection above. The skew may be "S"-shaped or "Z"-shaped.



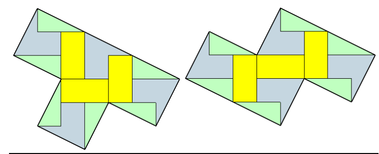
That makes a total of 17 target shapes, to be known as the **POLYOMINOES,** sometimes spelled "polyominos". The task is to find a group of the smallest number of puzzle pieces that can make each of these 17 shapes.

First Attempts

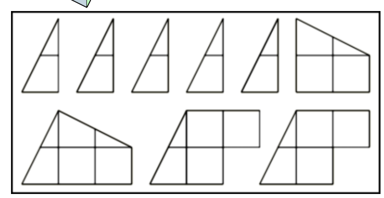
A group of 16 triangles and 4 squares can make all of the pentominoes and tetrominoes. For the tetrominoes, there are 10 triangles around the perimeter and the other pieces fill the interior. But 20 pieces is a big number. Can it be reduced?



By combining some building blocks in pairs, this number 20 can be reduced to about 13, as shown for the T and S tetrominoes. But it is difficult to get any lower using this method.

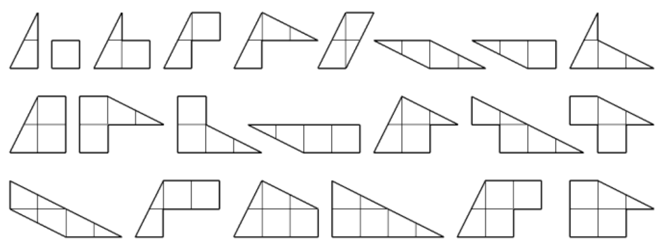


A research group at the Politecnico di Torino published this group of pieces that can make all 17 polyominoes. Web reference:
`http://www.iread.it/Poly/tepe_diss_en.php`
It has only nine pieces, five of them being the basic triangles.

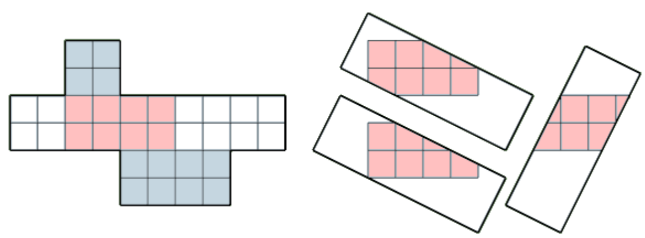


Analysis of the Problem

It is not easy to identify the most appropriate puzzle piece shapes to try. The unit square and small triangle can be glued together in many ways to make plausible puzzle pieces, ranging in area from one unit to six. Then each group of pieces that look promising must be tested against all 17 of the target shapes. A very tedious process!



No puzzle piece can be larger than 6 units. In the left diagram a white I pentomino is laid over a grey W pentomino. The pink rectangular area common to both has an area of 8 units. But when this is overlaid by an I tetromino, the pink area is reduced in one of the three ways as shown.

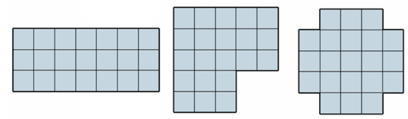


There are about 30 ways of combining small triangles and squares to make plausible puzzle pieces. These can be assembled into hundreds of groups and each group has to be tested to see if it can make all 17 of the target polyomino shapes. Obviously a computer is needed to do some of the computational “heavy lifting”, and a program such as "Burr Tools" is called for. But, even then, a lot of human intervention is required, together with a rather sophisticated search procedure.

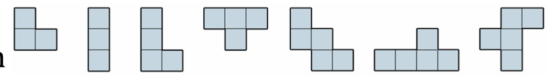
Using Burr Tools to Solve Puzzles

To illustrate how this can be done, we will set up an example problem and use Burr Tools to help in the search for a solution. This worked example is probably simple enough to be solved without a computer, but the real application, involving all the polyominoes, needs both Burr Tools and some new supporting computer programs, both for data preparation and for analysis. These will not be described in detail, but their main features should be fairly easy to follow.

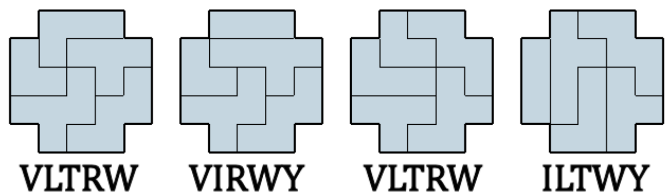
In this example there are three target shapes: "block", "gamma" and "cross". All have the same area, 21 units.



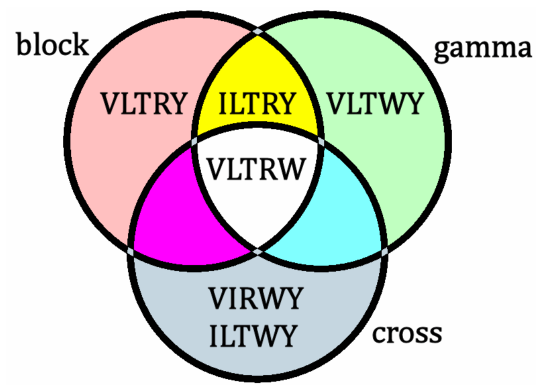
And there is a collection of pieces that is more than enough to cover that area. The pieces are called V, I, L, T, W, Y and R. They have a combined area of 29 units. So, any solution will use some of the puzzle pieces, but not all of them.



Here are some of the solutions for the "cross" target shape. The group of pieces for the first and third one is VLTRW, and the groups for the others are shown. But although there are many solutions, there are only three different groups.



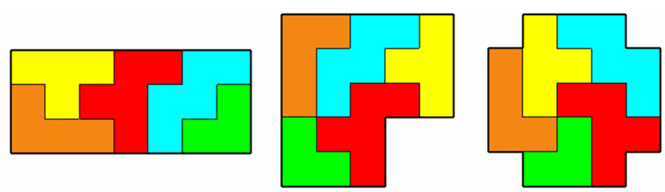
The shape "block" has several possible groups of pieces:
VLTRY, VLTRW, ILTRY.
And for "gamma", the groups are:
VLTRW, ILTRY, VLTWY.



The task is to identify a single group of pieces that can make ALL THREE of the target shapes. The results can be drawn on a Venn diagram. Each of the circles is the set of groups of pieces that can make one particular shape.

So the common group is VLTRW, shown in the centre:

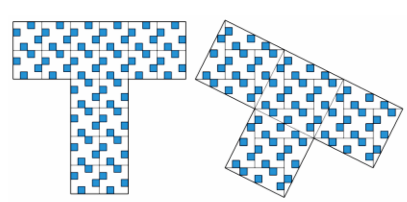
And here it can be seen that, indeed, the group of pieces VLTRW can make each of the three target shapes.



Adapting Burr Tools for the [1, 2, √5] triangle
Burr Tools usually deals with squares or equilateral triangles. So a modification is needed for the [1, 2, √5] triangle. These diagrams show how this was done. Everything was quadrupled in size and the unit square and triangle were represented like this:

It is a laborious and error-prone task to get the coding of all the puzzle pieces and polyomino shapes exactly right.

So a small shape definition program, written in Python, was used to help prepare the puzzle data. These diagrams show the T pentomino and the T tetromino.



The xmpuzzle file format

Burr Tools uses an XML format to describe the composition of puzzle pieces and target shapes. Everything is embedded in a file with an **`xmpuzzle`** extension. Usually this file is "zipped" before being written to disc. The **`xmpuzzle`** file also shows the details of the puzzle and, if some solutions have been found, these are embedded in the file as well before it is saved back to disc.

So it is possible to unzip these files, make some changes manually, and present them again to Burr Tools for further computation.

The structure of an **`xmpuzzle`** file is a bit complicated, but here is a condensed version of the file for the "block gamma cross" example puzzle above. Some of the XML has been removed for brevity, as well as parts of those sections which have a lot of repetition.

The main sections are these:

- A definition of all the shapes, both target shapes and puzzle pieces (in yellow).
- Indication of which shape is the target.
- Choice of the pieces to be used and how many of each (in blue). This might be a fixed number or a range of numbers.
- If the program has been run, the solutions are written back into the file (in pink).

```
<?xml version="1.0"?> <puzzle version="2"> <gridType type="0"/> <colors/>
 <shapes>
  <voxel x="3" y="7" z="1" type="0">#####################</voxel>
  <voxel x="5" y="5" z="1" type="0">###__###__##############</voxel>
          --(more lines like this)--
  <voxel x="4" y="2" z="1" type="0">####__#_</voxel>
  <voxel x="3" y="3" z="1" type="0">_#_##__##</voxel>
 </shapes>
 <problems>
  <problem state="2" assemblies="6" solutions="0" time="0">
   <shapes>
    <shape id="3" min="0" max="1"/>
    <shape id="4" min="0" max="1"/>
         --(more lines like this)--
    <shape id="9" min="0" max="1"/>
   </shapes>
   <result id="0"/>  <bitmap/>
   <solutions>   <solution>
     <assembly>0 0 0 0 x 2 6 0 10 1 0 0 18 x 0 3 0 18 2 4 0 10</assembly>
     <assembly>0 0 0 0 x 2 6 0 10 1 4 0 16 2 2 0 10 x 0 4 0 20</assembly>
        --(more lines like this)--
     <assembly>x 0 1 0 0 0 6 0 20 0 1 0 2 x 2 4 0 22 0 3 0 0</assembly>
    </solution>  </solutions>  </problem>  </problems>  <comment/>
</puzzle>
```

Without going into all the details, it can be seen that the **`<voxel>`** sections (in yellow) are describing the shapes of the three target shapes and the seven puzzle pieces. The symbols **`"#"`** and **`"_"`** (sharp and underscore) represent filled and empty cells in a matrix.

The **`<solutions>`** section (in pink) describes the solutions that have been found. The numbers come in sets of four (with a simple **`"x"`** if a piece is not being used). So it is a straightforward programming task to do some string processing and identify the group of pieces that has been used for any one solution. For our purposes the group is more relevant than the full solution.

Putting it all together
Having described the various logical components of the investigation, let's have a look at all the procedures involved. Two sorts of experiments were done:

- A search for a group of pieces, with no duplicates, which can make each of the 17 polyominoes.
- A search for the smallest number of pieces, this time allowing any number of duplicates.

Suppose we have a **BOX** of pieces, possibly containing some duplicates, and we present it to Burr Tools together with just one of the 17 polyomino shapes. We will then get a **COLLECTION** of **SOLUTIONS** for that particular shape. This process is then repeated 16 more times to cover all the target shapes.

The choice of the pieces in the box is a matter for the human investigator. If there are too few pieces, or if they are badly chosen, there may not be enough variety for a complete solution to emerge. But, if there are too many pieces, the computing complexity may be too great.

Within the collection of solutions for one target shape, there may be a group of pieces that is used for more than one solution. We are more interested in the **groups** than in the **solutions**. So the collection of groups needs to be reduced to a **SET** with no repetitions. The word **set** is being used in its mathematical sense of a collection of objects, no two of which are identical. Each object in the set is a group of puzzle pieces, usually between 8 and 11 pieces in any one group.

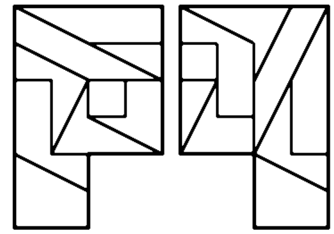
In this way, we will get 17 sets of groups of pieces and we need to find a group that is present in all 17 sets. So we make an “intersection” of the 17 sets, looking for the one element that is present in all of them. This way we hope to get a group of pieces that can be used to make all 17 polyominoes.

There may be several such groups, or there may be none at all. This would mean that, for that particular selection of pieces in the box, there is no group that can make all of the 17 target shapes. So it may be necessary to adjust the box of plausible pieces and try again. That might mean doubling up on a few pieces, including or inventing new ones and leaving out others.

Mirror Image Target Shapes
And now for a small complication. The puzzle pieces that we are working with are all non-symmetrical, except for the unit square. But the only angles in the polyomino target shapes are right angles.

So it is not possible to turn over just one puzzle piece and leave the others as they were. Therefore, unlike most put-together puzzles, it is NOT permitted to turn over any piece, unless all of them are turned over.

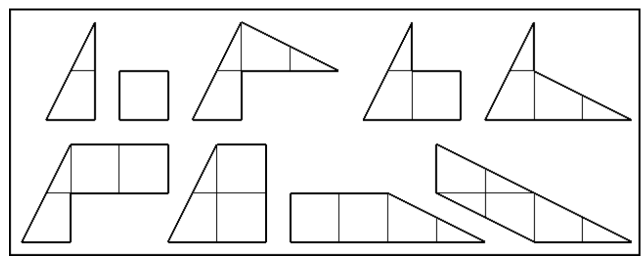
This means that if there is a solution to one target shape, the P pentomino for example, there will be quite a different solution for its mirror image, as seen in this nine-piece assembly.



Or there could be a polyomino which has a perfectly valid solution, but its mirror image has no solution at all.

So, although there are just 12 pentominoes and 5 tetrominoes, there are actually 8 more shapes to be considered if the mirror images of the non-symmetric polyominoes are included. This means that we can, for example, include the "skew tetromino" shape if there is a solution for its "Z" configuration, even though there is no solution for its "S" configuration.

A Nine-Piece Group with no Duplicate Pieces
Here is a group of 9 pieces that are all different. It can make all the 17 polyominoes.

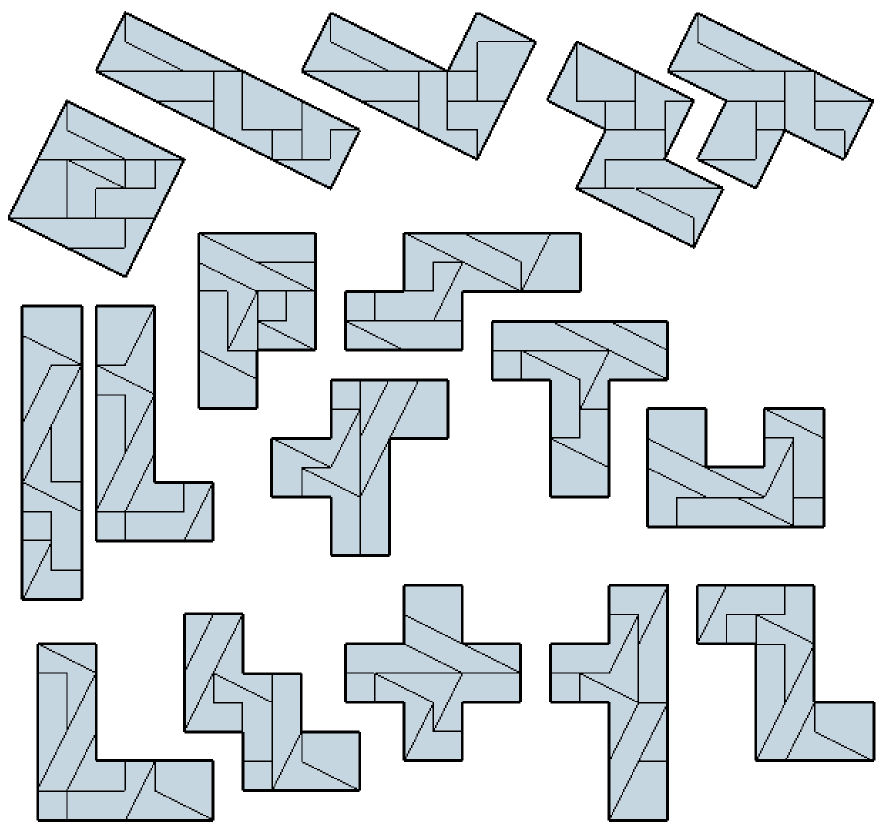


Using this nine-piece group, there are several solutions for all the polyominoes and some, but not all, of their mirror images.

This diagram gives an indication of the complexity of the solutions in this project.

The skew tetromino has been drawn in its "Z" configuration. That is because there is not a solution in its "S" configuration.

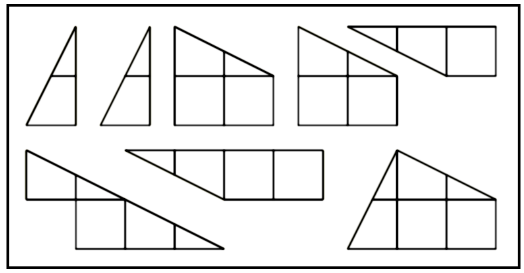
There are also no solutions for the mirror image shapes of the L tetromino and the R, S, and Z pentominoes.

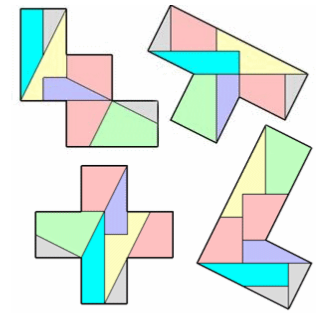


Eight-Piece Groups

So far, three groups of eight pieces have been found, one of which is shown here.

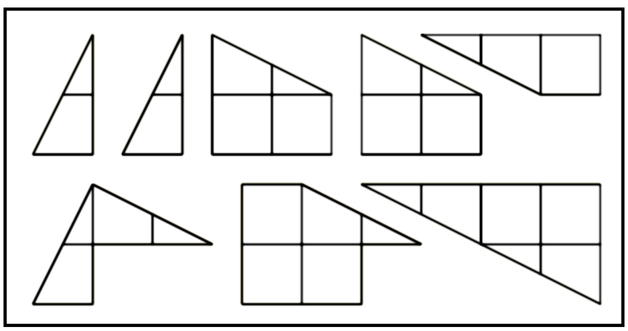
Some solutions using this group are shown at far right.

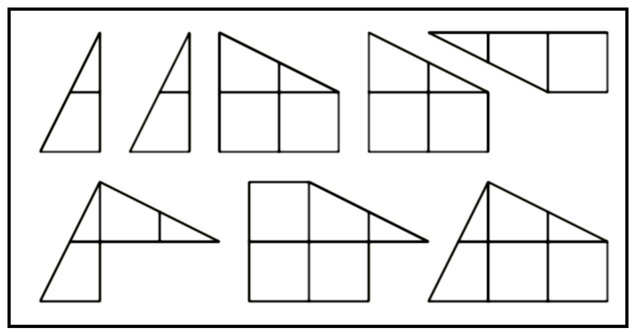




This was my Exchange Gift at the Gathering for Gardner in Atlanta in 2018. It has quite a small number of solutions for all of the tetrominoes and pentominoes and nearly all of the mirror image shapes, too. So it is a challenging collection of puzzles. It can't make the "Z" configuration of the skew tetromino, so the total number of puzzles is 24, not 25.

And here are two more groups of eight pieces which can make the 17 polyominoes. In each case the top row of five pieces is the same as in the group above.

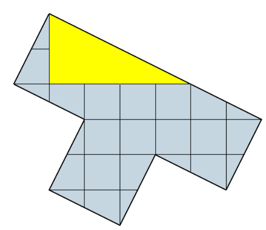




Conclusions and Opportunities for Further Work

The procedure to find these groups of pieces was far from straightforward. The number of intermediate solutions found by Burr Tools was huge, and the computer frequently ran out of storage and just stopped.

Sometimes a consideration of the target shapes demonstrated that a particular puzzle piece could be used once but not twice. The long edge of the big triangle can't, for example, fit twice into the perimeter of the T tetromino.



So it has not been possible to do an exhaustive search for the very best groups of pieces that can make all 17 polyominoes. But the difficulty of finding a group of just eight pieces suggests strongly that no seven-piece group exists.

The solution in nine pieces may not be the only one, and it is possible that there is an eight-piece group with all the pieces different, which is still waiting to be found. Bigger computer needed!

Please email me with comments, discoveries and suggestions.