DIFFERENCE DICE

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1. Sum Dice

George Sicherman is one of the many nonprofessional mathematicians whose work was highlighted by Martin Gardner [4]. Sicherman is credited with finding alternative positive integer labels for two six-sided dice whose sums and frequencies match two standard labeled six-sided dice as shown in Table 1.

Table 1. Two pairs of labeled six-sided dice, both of whose sums are one 2, two 3s, three 4s, four 5s, five 6s, six 7s, five 8s, four 9s, three 10s, two 11s, and one 12.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

+	1	3	4	5	6	8
1	2	4	5	6	7	9
2	3	5	6	7	8	10
2	3	5	6	7	8	10
3	4	6	7	8	9	11
3	4	6	7	8	9	11
4	5	7	8	9	10	12

An elegant approach to this problem uses generating functions or enumerators: Represent the sums of two standard labeled six-sided dice as $(x+\cdots+x^6)^2$. Sicherman's "crazy dice" arise from a different way of breaking the polynomial into factors. In particular,

$$\begin{aligned}
&(x^1+x^2+x^3+x^4+x^5+x^6)^2\\
&= x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}\\
&= x^2\,(1+x)^2\,(1-x+x^2)^2\,(1+x+x^2)^2\\
&= (x(1+x)(1+x+x^2))\,(x(1+x)(1-x+x^2)^2(1+x+x^2))\\
&= (x+2x^2+2x^3+x^4)\,(x+x^3+x^4+x^5+x^6+x^8)
\end{aligned}$$

For various generalizations, see [1, 2, 3]. Here are the criteria for a polynomial $d_i(x)$ to correspond to a die with positive integer labels.

- $d_i(0)=0$ as a nonzero constant term cx^0 would indicate c faces labeled 0.
- $d_i(1)=6$ so that the sum of the coefficients matches the number of faces of the die.
- To match a count, the coefficients must be nonnegative (although factors along the way can have negative coefficients, as above).

In the example above, the flexibility in grouping factors comes from the fact that evaluating $1-x+x^2$ at $x=1$ gives 1.

2. Difference Dice

What if the sum operation is replaced by the difference? Since the dice are indistinguishable, the absolute value of the difference is a reasonable statistic to consider, as shown in in Table 2.

Table 2. Two standard labeled six-sided dice have differences six 0s, ten 1s, eight 2s, six 3s, four 4s, and two 5s.

$-$	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Are there other labelings of two six-sided dice with the same differences and frequencies? Well, increasing all labels by a constant does not change differences, so two dice labeled $\{2,3,4,5,6,7\}$ have the same difference table. Requiring the minimal label to be 1 therefore picks one representative from an infinite family of solutions.

To look for substantially different labelings, we can adapt the algebraic approach for sums:

$$\begin{aligned}
&\left(x^1+x^2+x^3+x^4+x^5+x^6\right)\left(x^{-1}+x^{-2}+x^{-3}+x^{-4}+x^{-5}+x^{-6}\right)\\
&= x^{-5}+2x^{-4}+3x^{-3}+4x^{-2}+5x^{-1}+6+5x+4x^2+3x^3+2x^4+x^5\\
&= \left(x^1+x^2+x^3+x^4+x^5+x^6\right)\left(\frac{x^1+x^2+x^3+x^4+x^5+x^6}{x^7}\right)\\
&= \frac{\left(x^1+x^2+x^3+x^4+x^5+x^6\right)^2}{x^7}\\
&= \frac{x^2\,(1+x)^2\,(1-x+x^2)^2\,(1+x+x^2)^2}{x^7}\\
&= \left(x\,(1+x)\,(1-x+x^2)^2\,(1+x+x^2)\right)\left(\frac{x\,(1+x)\,(1+x+x^2)}{x^7}\right)\\
&= \left(x^1+x^3+x^4+x^5+x^6+x^8\right)\left(x^{-3}+2x^{-4}+2x^{-5}+x^{-6}\right)
\end{aligned}$$

which leads to labeled six-sided dice shown in Table 3.

Table 3. Another pair of six-sided dice with differences six 0s, ten 1s, eight 2s, six 3s, four 4s, and two 5s.

$-$	1	3	4	5	6	8
3	2	0	1	2	3	5
4	3	1	0	1	2	4
4	3	1	0	1	2	4
5	4	2	1	0	1	3
5	4	2	1	0	1	3
6	5	3	2	1	0	2

(Note that it would not have worked to divide the other polynomial by x^7 as that would result in a mix of positive and negative exponents.) These dice are based on the same factorization as the Sicherman (sum) dice, so one might suspect that this is essentially the only possible alternative solution.

But Table 4 gives another substantially different solution.

TABLE 4. Yet another pair of six-sided dice with differences six 0s, ten 1s, eight 2s, six 3s, four 4s, and two 5s.

−	1	2	2	3	3	6
1	0	1	1	2	2	5
2	1	0	0	1	1	4
3	2	1	1	0	0	3
4	3	2	2	1	1	2
5	4	3	3	2	2	1
6	5	4	4	3	3	0

What is happening algebraically with these?

$$\begin{aligned}&(x^1+x^2+x^3+x^4+x^5+x^6)\,(x^{-1}+2x^{-2}+2x^{-3}+x^{-6})\\&=x^{-5}+x^{-4}+x^{-3}+3x^{-2}+5x^{-1}+6+5x+5x^2+5x^3+3x^4+x^5\end{aligned}$$

Unlike the symmetric situation with standard dice, where x^i and x^{-i} have the same coefficient, this pair has, for example, $3x^{-2}$ and $5x^2$. Since our operation is the absolute value of the difference, it is too restrictive to require that pairs of dice correspond to

$$x^{-5}+2x^{-4}+3x^{-3}+4x^{-2}+5x^{-1}+6+5x+4x^2+3x^3+2x^4+x^5.$$

Instead, writing $ax^{-i}+bx^i$ as $(a+b)x^{\pm i}$, the three pairs of dice all correspond to

$$6+10x^{\pm 1}+8x^{\pm 2}+6x^{\pm 3}+4x^{\pm 4}+2x^{\pm 5}.$$

In fact, many others do too. An exhaustive computer search is feasible for finding all solutions matching standard six-sided dice, but a better theoretical understanding is required to handle larger cases. This leads to our closing plea.

Question: What tools allow one to find all "factorizations" of expressions such as $6+10x^{\pm 1}+8x^{\pm 2}+6x^{\pm 3}+4x^{\pm 4}+2x^{\pm 5}$?

REFERENCES

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