

# Nontransitive Dice for Three Players

James Grime

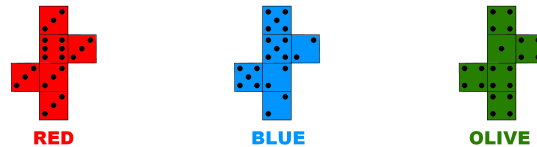
## 1 Three Unusual Dice

Here is a game you can play with a friend. It is a game for two players, with a set of three dice. These dice are not typical dice however, because instead of having the values 1 to 6, they display various unusual values.

The game is simple: Each player picks a die. The two dice are then rolled together and whoever gets the highest value wins.

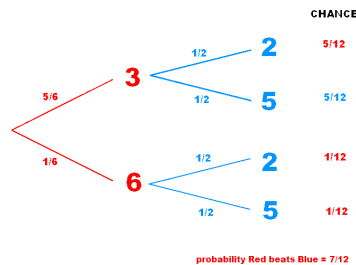
The game seems fair enough. Yet, in a game of, say, ten rolls, you will always be able to pick a die with a better chance of winning - no matter which die your friend chooses. And you can make these dice at home right now.

Here is the set of three special dice:



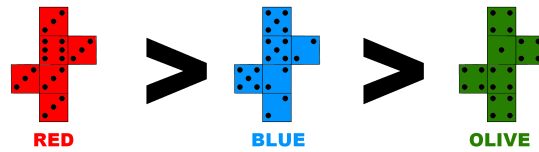
We say A beats B if the probability of die A beating die B is greater than 50%.

It's simple to show that the Red die beats the Blue die by way of a tree diagram:



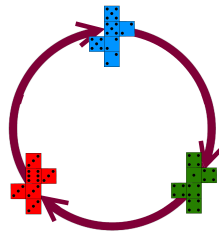
From the diagram we see Red beats Blue with a probability of  $\frac{7}{12}$ . This is greater than 50% so Red is the better choice here.

Similarly, it can be shown that Blue beats Olive with a probability of  $\frac{7}{12}$ . So we can set up a winning chain where Red beats Blue, and Blue beats Olive.



Using this information it would be perfectly reasonable to expect, therefore, that Red beats Olive. If this is true we call the dice ‘transitive’.

However, this is not the case. In fact, bizarrely, Olive beats Red with a probability of  $\frac{25}{36}$ . This means the winning chain is a circle - like a game of ‘Rock, Paper, Scissors’.



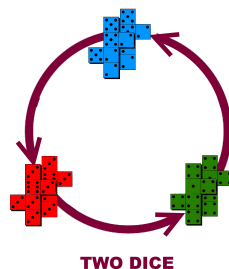
This is what makes the game so tricky because, as long as you let your opponent pick first, you will always be able to pick a die with a better chance of winning.

## 2 Double Whammy

After a few defeats your friend may have become suspicious, but all is not lost. Once you’ve explained how the dice beat each other in a circle, challenge your friend to one more game.

This time you will choose first, in which case your opponent should be able to pick a die with a better chance of winning. But let’s increase the stakes, and increase the number of dice. This time each player rolls two of his chosen die, so that the player with the highest total wins.

Maybe using two dice means your opponent has just doubled their chances of winning. But not so because, amazingly, with two dice the order of the chain flips!

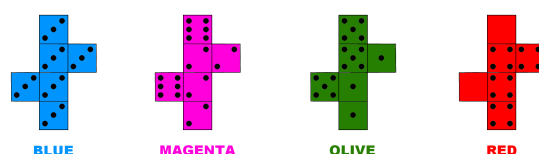


In other words, the chain reverses so the circle of victory now becomes a circle of defeat - allowing you to win the game again!

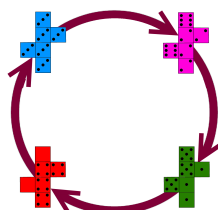
### 3 Efron Dice

The paradoxical nature of nontransitive dice goes back to 1959 and to the Polish mathematicians Hugo Steinhaus and Stanislaw Trybula [4].

However, the remarkable reversing property is not true for all sets of nontransitive dice. For example, here is a set of four nontransitive dice introduced by Martin Gardner in his column Mathematical Games in 1970 [2]. This set is known as ‘Efron Dice’ and was invented by the American statistician Brad Efron:



Here, the dice form a circle where Blue beats Magenta, Magenta beats Olive, Olive beats Red, and Red beats Blue, and they each do so with a probability of  $\frac{2}{3}$ .



Usiskin and Trybula independently showed [7], [6] that it was always possible to set up a nontransitive system of  $m$   $n$ -sided dice, and showed that the weakest winning probability has a bound. It is not possible for all winning probabilities to exceed this bound, but it is possible for all winning probabilities to be greater than, or equal to, this bound.

For six-sided dice, the set of three dice above achieve this bound. Using a different number of sides the greatest bound for three dice is the Golden Ratio  $\varphi = 0.618\dots$ . This theoretical bound increases as the number of dice increases, and converges to  $\frac{3}{4}$ .

Efron Dice achieve the bound for four dice of  $\frac{2}{3}$ . Unfortunately, they do not possess remarkable reversing property when you double the number of dice - while some of the probabilities reverse, others do not.

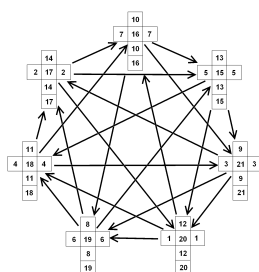
It is said the billionaire American investor Warren Buffett is a fan of nontransitive dice. When he challenged his friend Bill Gates to a game, with a set of Efron dice, Bill became suspicious and insisted Warren choose

first. Maybe if Warren had chosen a set with a reversing property he could have beaten Gates - he would just need to announce if they were playing a one die or two dice version of the game after they had both chosen.

## 4 Three Player Games

I wanted to know if it was possible to extend the idea of nontransitive dice to make a three player game - a set of dice where two of your friends may pick a die each, yet you can pick a die that has a better chance of beating both opponents - at the same time!

It turns out there is a way. The Dutch puzzle inventor M. Oskar van Deventer came up with a set of seven nontransitive dice, with values from 1 to 21. Here two opponents may each choose a die from the set of seven, and yet there will be a third die with a better chance of beating each of them. The probabilities are remarkably symmetric with each arrow on the diagram illustrating a probability of  $\frac{5}{9}$ .



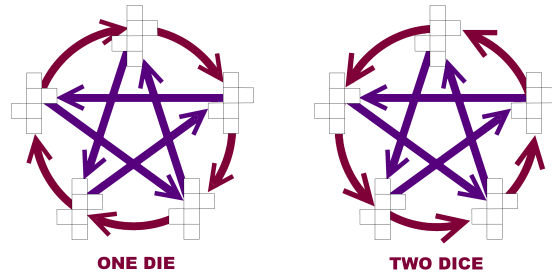
This means we can play two games simultaneously, however beating both players at the same time is still a challenge. The probability of doing so stands at around 39%.

This set of seven dice form a complete directed graph. In the same way, a four player game would require 19 dice. A direct construction of this set was not known until 2016 when Angel and Davis devised a direct construction for any tournament of any number of dice [1].

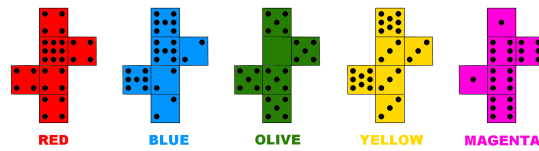
However, I began to wonder if it was possible to exploit the reversing property of some nontransitive dice to design a slightly different three player game, one that uses fewer than seven dice.

## 5 Grime Dice

My idea for a three player game required a set of five dice that contained two nontransitive chains. When the dice were doubled one chain would remain in the same order, while the second chain would reverse. This way, choosing a one or two dice version of the game will allow you to play two opponents at the same time, no matter which dice they pick.

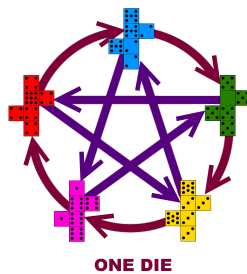


After a small amount of trial and error, I devised the following set of five nontransitive dice:



These dice appeared to be the best set of five I could find. I have written about them before and they became known as Grime Dice.

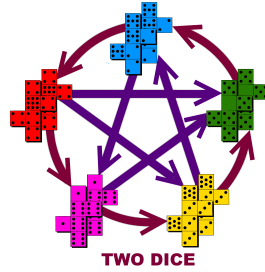
For one die games we get the following chains:



All winning probabilities here are equal to, or greater than,  $\frac{5}{9}$  with an average winning probability of 63% - I will leave the calculations to the interested reader. Notice the first chain is ordered alphabetically, while the second chain is ordered by word-length.

You can also find nontransitive subsets of dice. For example, the Red, Blue and Olive dice are a copy of the original set of three nontransitive dice that I describe above, complete with the same winning probabilities and reversing property.

For two dice games we get the following chains:



An unfortunate consequence of Red, Blue and Olive having the reversing property is that, when we double the dice, the first chain (the outside circle) reverses order, while the second chain (the inside pentagram) stays the same - with one exception.

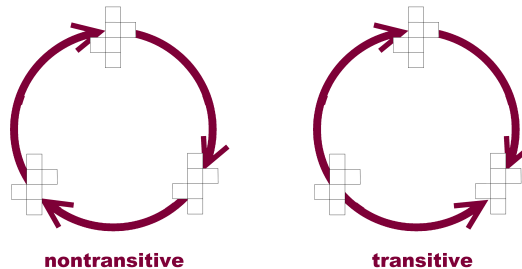
However, the probability of this exception is very close to 50% (it's  $\frac{625}{1296}$ ). Meanwhile, the average of all other winning probabilities is 62% - much higher than Oskar Dice - and so, in practice, the three player game still works.

It's quite nice that this set of five contained three dice with their own reversing property. However, I admit, the exception continued to niggle at me. I wanted to know if there was a set of five nontransitive dice, with the desired properties, with no exceptions - or was this set really as close as we could get.

## 6 Finding a new set of Grime Dice

I enlisted the help of a computer, and the invaluable help of my friend Brian Pollock, to search for sets of five nontransitive dice. The computational challenge of working out all sets of five dice, and their chains, was a large one - so we devised a test.

Three dice can either form a diagram with all three arrows in the same direction, which we call a nontransitive chain, or with only two arrows in the same direction, which we call a transitive chain.



We wanted to create a set of five nontransitive dice, with two nontransitive chains, such that, when doubled, one chain stays the same, and the other chain reverses order.

This will mean that, for any subset of three dice, if they form a nontransitive chain singly, they will form a transitive chain when doubled. Alternatively, if they form a transitive chain singly, they will form a

nontransitive chain when doubled. If a chain remains transitive or nontransitive when the dice are doubled, then we say the set has failed the test.

There are 10 subsets of three dice from a set of five. Each subset needs to pass the test. Furthermore, if all subsets pass the test, we have found a valid set of five dice with the desired properties.

The application of this test allowed us to reject sets without the desired property with less calculation.

In the first instance, we only considered dice using the values 0 to 9. Sets of dice that allow draws would be rather unsatisfactory, so after excluding draws, no set of five dice passed the test.

Only a few sets of four dice passed the test, which simply turned out to be the original Grime Dice with one of the dice missing. This proved that Grime Dice really are the best set of five dice using the values 0 to 9, without draws.

## 7 Dice with higher values

Naturally, the next thing to try were dice with higher values. Keeping the criteria of no draws, the first success found used the values 0 to 13.

A : 4 4 4 4 4 9

B : 2 2 2 7 7 12

C : 0 5 5 5 5 10

D : 3 3 3 3 8 13

E : 1 1 6 6 6 11

There were two such sets using the values 0 to 13, with the second set being only a slight variation of the above. These were also the only sets of five with the desired properties that uses consecutive numbers.

I was delighted with this success, but the winning probabilities for this set were weaker than Grime Dice. The average winning probability is about 59%, lower than Grime Dice. So we continued our search, to find a set with stronger winning probabilities.

The winning probabilities slowly increased as we increased the values on the dice. Here is one of the strongest sets of five dice using the values 0 to 17:

A : 4 4 8 8 8 17

B : 2 2 2 15 15 15

C : 0 9 9 9 9 9

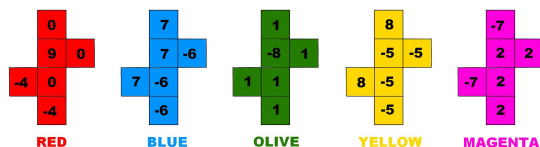
D : 3 3 3 3 16 16

E : 1 1 10 10 10 10

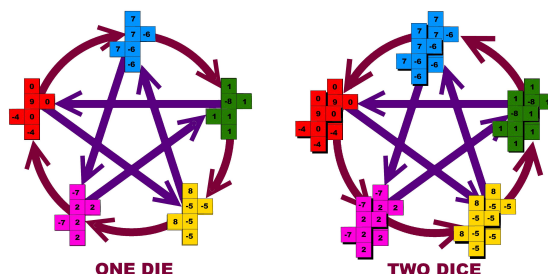
Increasing the dice values after this point did nothing to improve the winning probabilities. Since the

numbers are no longer consecutive there is enough space for the values to change without changing the winning probabilities, meaning this set can appear repeatedly in slightly different forms. The investigation for better sets had plateaued.

For aesthetic reasons, I decided to subtract 8 from all sides of the above dice, making a set of New Grime Dice using the values from -8 to 9:



Like the original Grime Dice, this set makes two nontransitive chains, one with the colours listed alphabetically, the other with the colours listed by word length. When doubled, the alphabetical chain remains in the same order, while the chain ordered by word length flips.



In single dice games, New Grime Dice have the exact same winning probabilities as Original Grime Dice. When the dice are doubled, New Grime are generally slightly weaker than Original Grime Dice. The average winning probability for New Grime Dice is 60.4%, that's 0.7% lower than Original Grime Dice. Crucially however, all winning probabilities are now over 50%, allowing for a true three player game as follows:

Invite two opponents to pick a die each, but do not volunteer whether you are playing a one die or two dice version of the game. No matter which dice your opponents pick, you will always be able to pick a die to beat each opponent. If your opponents pick two dice that are consecutive alphabetically, play the one die version of the game. If your opponents pick two dice that are consecutive by word-length, use the two dice version of the game.

## 8 A Gambling Game

But, can we expect to beat the two other players at the same time? Well, we have certainly improved the odds, with the average probability of beating both opponents now standing around 44% - a 5% improvement over Oskar Dice. So, if the odds of beating two players isn't over 50% then how do we win? Consider the following gambling game:



Challenge two friends to a dice game, where you will play your two opponents at the same time. If you lose you will give your opponent 1. If you win your opponent gives you \$1. So, if you beat both players at the same time you win \$2; if you lose to both players you lose \$2; and if you beat one player but not the other then your net loss is zero. You and your friends decide to play a game of 100 rolls.

If the dice were fair then each player will expect to win zero - each player wins half the time and loses half the time.

However, with Oskar Dice, you should expect to beat both players 39% of the time, and lose to both players 28% of the time, which will give you a net profit of \$22.

But even better, with New Grime Dice, you should expect to beat both players 44.1% of the time, but only lose to both players 23.6% of the time, giving you an average net profit closer to \$41! (And possibly the loss of two former friends).

I invite you to try out these games yourself, and enjoy your successes and failures!

## References

- [1] L. Angel, M. Davis, A Direct Construction of Non-Transitive Dice Sets. *Journal of Combinatorial Designs*. (2016) 10.1002/jcd.21563.
- [2] M. Gardner, Mathematical games: The paradox of the nontransitive dice and the elusive principle of indifference, *Sci. Amer.* **223** (Dec. 1970) 110114.
- [3] R. P. Savage Jr., The paradox of nontransitive dice, *Amer. Math. Monthly* **101** (1994) 429436, <http://dx.doi.org/10.2307/2974903>.
- [4] H. Steinhaus, S. Trybula, On a paradox in applied probabilities, *Bull. Acad. Polon. Sci.* **7** (1959) 6769.
- [5] S. Trybula, On the paradox of three random variables, *Zastos. Mat.* **5** (1960/1961) 331332.
- [6] S. Trybula, On the paradox of n random variables, *Zastos. Mat.* **8** (1965), 143-154.
- [7] Z. Usiskin, Max-Min Probabilities in the Voting Paradox, *Annal. Mat. Stat.* **35** (June 1964), 857-862.