Playing nice with a crooked coin

1 A motivating problem

Two kids have found a coin and want a fair way of deciding who gets to keep it, by tossing it a finite number of times. Let p be the probability of the coin landing heads. We may assume by symmetry that $0 < p \leq \frac{1}{2}$. Suppose the coin is fair, that is, $p = \frac{1}{2}$. Clearly one toss will suffice.

Suppose the coin is crooked. We may toss the coin twice. A head followed by a tail means that the first kid gets the coin, while a tail followed by a head means that the second kid gets the coin. If the coin lands heads both times or tails both times, the process is repeated. This is clearly fair. However, if some super-being is having fun with the kids, they may be tossing heads until the end of time. Thus this is not a solution as it violates the condition that a fair decision must be reached within a finite number of tosses.

If $0 < p < \frac{1}{2}$, the task is not always possible, but there are infinitely many values of p for which workable protocols exist. Suppose we toss the coin twice. Two tails means that the first kid gets the coin. Otherwise the other kid gets it. To make this fair, we need $(1-p)^2 = \frac{1}{2}$. Hence $1 - p = \frac{1}{\sqrt{2}}$ and $p = 1 - \frac{1}{\sqrt{2}}$.

For another possible value, suppose we toss the coin three times. The first kid gets the coin if and only if it lands tails all three times. Then $(1-p)^3 = \frac{1}{2}$ and $p = 1 - \frac{1}{\sqrt[3]{2}}$. It is clear that $p = 1 - \frac{1}{\sqrt[n]{2}}$ works for any positive integer n.

2 A second example, introducing Pascal's triangle

Let's look at four kids now. In this example we give an idea of a general technique to produce solutions using Pascal's triangle. We present three possible solutions.

1. The obvious value is $p = \frac{1}{2}$, and the coin needs to be tossed only twice. Flipping the coin twice yields the possible results of HH, HT, TH, TT of equal probability. Assigning each result to a person provides a fair game.

2. Suppose the coin is not fair. Let $q = 1 - p$. Tossing it six times, we have Let $q = 1 - p$. Suppose we flip the coin six times. Then,

$$1 = (p+q)^6$$

so after expanding we have,

$$1 = p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6\,. \tag{1}$$

Next we collect terms divisible by 3 together

$$1 = 3(2p^5q + 5p^4q^2 + 6p^3q^3 + 5p^2q^4 + 2p^5q) + p^6 + 2p^3q^3 + q^6\,.$$

For each of the three copies of the outcomes given by $2p^5q + 5p^4q^2 + 6p^3q^3 + 5p^2q^4 + 2p^5q$ we can assign to one person. This way we guarantee each of them will have the same probability of winning. Therefore it suffices to make sure the final person has the same probability as the others, ie we want to find find p and q such that

$$\frac{1}{4} = p^6 + 2p^3q^3 + q^6$$

or

$$\frac{1}{4} = (p^3 + q^3)^2 \,.$$

Defining $0 < r < \frac{1}{2}$ as $p = \frac{1}{2} - r$ yields

$$\frac{1}{4} = \left(\left(\frac{1}{2} - r\right)^3 + \left(\frac{1}{2} + r\right)^3 \right)^2$$

$$\frac{1}{4} = \left(\frac{1}{4} + 3r^2\right)^2 ,$$

which lets us solve $r = \frac{1}{\sqrt{12}}$.

3. Similarly tossing a different crooked coins nine times, $1 = ((\frac{1}{2} + r) + (\frac{1}{2} - r))^9$ is the sum of

$$\left(\frac{1}{2} + r\right)^9 + 3\left(\frac{1}{2} + r\right)^6 \left(\frac{1}{2} - r\right)^3 + 3\left(\frac{1}{2} + r\right)^3 \left(\frac{1}{2} - r\right)^6 + \left(\frac{1}{2} - r\right)^9$$

and other terms whose coefficients are all multiples of 3. So we set

$$\frac{1}{4} = \left(\left(\frac{1}{2} + r\right)^3 + \left(\frac{1}{2} - r\right)^3 \right)^3 = \left(\frac{1}{4} + 3r^2\right)^3 .$$

It follows that $\frac{1}{4} + 3r^2 = \frac{1}{\sqrt[3]{4}}$ so that $r = \sqrt{\frac{4 - \sqrt[3]{4}}{12\sqrt[3]{4}}}$.

In summary, three possible values are $p = \frac{1}{2}$, $p = \frac{1}{2} - \frac{1}{\sqrt{12}}$ and $p = \frac{1}{2} - \sqrt{\frac{4-\sqrt[3]{4}}{12\sqrt[3]{4}}}$.

Remark.
Where did the inspiration to check 6 flips and 9 flips come from? The solution above is based on Pascal's Triangle, involving a subtraction of the 2nd row from the 6th row, and a subtraction of the 3rd row from the 9th row. In particular, for the second solution ($p = \frac{1}{2} - \frac{1}{\sqrt{12}}$) the numbers in the 6th row not circled are already divisible by 3, and subtracting the circled numbers in the 2nd row from the 3rd yields numbers which are divisible by 3.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

In this sense, the 6th row "minus" the 2nd row yields only numbers divisible by 3. Because of this we are able to to rewrite (1) as a perfect square plus several terms with coefficients divisible by 3. This is our central strategy. Looking at more rows of Pascal's triangle we see this approach works again to get the probability $p = \frac{1}{2} - \sqrt{\frac{4-\sqrt[3]{4}}{12\sqrt[3]{4}}}$.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}$$

3 A third example, finding infinite solutions

What about an arbitrary number of kids? Call it n. If $n = 3$ we need to take the difference of two rows of Pascal's triangle to obtain only multiples of 2. This should be easier to do!

Let's consider the second row of Pascal's triangle,

$$1 = \left(\frac{1}{2} - r\right)^2 + 2\left(\frac{1}{2} - r\right)\left(\frac{1}{2} + r\right) + \left(\frac{1}{2} + r\right)^2,$$

so that removing the even terms and letting the leftover equal to 1/3 gives,

$$\frac{1}{3} = \left(\frac{1}{2} - r\right)^2 + \left(\frac{1}{2} + r\right)^2$$

so that,

$$-\frac{1}{12} = r^2$$

which has no real solutions. While we weren't lucky this time, this approach **does** work for flipping the coin more than twice!

Subtracting the fourth row from the second provides leaves of 2, which produces the probability of

$$p = \frac{1}{2} - \sqrt{\frac{1}{2\sqrt{3}} - \frac{1}{4}}.$$

Subtracting the sixth row from the third also leaves multiples of 2, this time producing the probability of

$$p = \frac{1}{2} - \sqrt{\frac{1}{2\sqrt[3]{3}} - \frac{1}{4}}.$$

It seems reasonable to conjecture that for 3 people, and with $k \geq 2$, if you flip the coin $2k$ times then setting the probability

$$p = \frac{1}{2} - \sqrt{\frac{1}{2\sqrt[k]{3}} - \frac{1}{4}}$$

will provide a setup for a fair game. It is worth noting that as $k \to \infty$ then $p \to 0$.

One can show this conjecture holds true as

$$\binom{2a}{2b} \equiv \binom{a}{b} \pmod{2},$$

and

$$\binom{2a}{2b+1} \equiv 0 \pmod{2}.$$

Similarly, for $n = 5$ people, one can apply a similar strategy. For any $k \geq 2$ we flip the coin $4k$ times and use a probability of,

$$p = \frac{1}{2} - \sqrt{\frac{1}{2\sqrt[2k]{5}} - \frac{1}{4}}.$$

This time the tricky part is proving the following lemma,

Lemma. *For positive integers a and b,*

(a)

$$\binom{4a}{2b+1} \equiv 0 \pmod{4} \quad \text{for } 0 \leq b \leq 2a-1,$$

(b)

$$\binom{4a}{2b} \equiv \binom{2a}{b} \pmod{4} \quad \text{for } 0 \leq b \leq 2a.$$

Proof. First we make repeated use of the recursive formula. Then for $0 \leq c \leq 4a$,

$$\begin{aligned}
\binom{4a}{c} &= \binom{4a-1}{c} + \binom{4a-1}{c-1} \\
&= \binom{4a-2}{c} + 2\binom{4a-2}{c-1} + \binom{4a-2}{c} \\
&= \binom{4a-3}{c} + 3\binom{4a-3}{c-1} + 3\binom{4a-3}{c-2} + \binom{4a-3}{c-3} \\
&= \binom{4a-4}{c} + 4\binom{4a-4}{c-1} + 6\binom{4a-4}{c-2} + 4\binom{4a-4}{c-3} + \binom{4a-4}{c-4}.
\end{aligned}$$

So we have shown

$$\binom{4a}{c} \equiv \binom{4(a-1)}{c} + 2\binom{4(a-1)}{c-2} + \binom{4(a-1)}{c-4} \pmod 4 . \tag{2}$$

We will prove (a) first. Notice it is trivially true for $b = 0$ (equivalently for $b = 2a - 1$), and for $b = 1$ (equivalently for $b = 2a - 2$) we have

$$\binom{4a}{3} = \frac{(4a)(4a-1)(4a-2)}{3 \cdot 2} \equiv 0 \pmod 4 .$$

We prove the rest by induction on a. The base case is easy to see. We may assume (a) is true for $a - 1$. By (2) with $c = 2b + 1$ we have,

$$\binom{4a}{2b+1} \equiv \binom{4(a-1)}{2b+1} + 2\binom{4(a-1)}{2b-1} + \binom{4(a-1)}{2(b-1)-1} \pmod 4$$

so by the inductive hypothesis,

$$\binom{4a}{2b+1} \equiv 0 + 2 \cdot 0 + 0 \equiv 0 \pmod 4 .$$

Next we prove (b). We have (b) is trivially true for $b = 0$ (equivalently $b = 2a$), and it is also true for $b = 1$ (equivalently for $b = 2a - 2$) as $\binom{2a}{1} = 2a$ and

$$\binom{4a}{2} = \frac{(4a)(4a-1)}{2} \equiv 2a(-1) \equiv 2a \pmod 4 .$$

For $b = 2$ (equivalently $b = 2a - 4$) we have,

$$\binom{4a}{4} = \frac{(4a)(4a-1)(2(2a-1))(4a-3)}{4 \cdot 3 \cdot 2} \equiv (-1)(-3)(-1)a(2a-1) \equiv a(2a-1) \pmod 4$$

and

$$\binom{2a}{2} = \frac{(2a)(2a-1)}{2} = a(2a-1) .$$

We can prove the rest by induction on a. The base case is again easy to see. We may assume (b) is true for $a - 1$. On one hand by (2) with $c = 2b$ we have,

$$\binom{4a}{2b+1} \equiv \binom{4(a-1)}{2b} + 2\binom{4(a-1)}{2(b-1)} + \binom{4(a-1)}{2(b-2)} \pmod 4$$

and on the other hand by the recursive formula,

$$\begin{aligned}\binom{2a}{b} &= \binom{2a-1}{b} + \binom{2a-1}{b-1} \\ &= \binom{2(a-1)}{b} + 2\binom{2(a-1)}{b-1} + \binom{2(a-1)}{b-2}\end{aligned}$$

and so $\binom{4a}{2b} \equiv \binom{2a}{b} \pmod 4$ by the inductive hypothesis. □

4 Finding more kids

Let us first give some general rules. Suppose we have a protocol which works for n kids and n has a non-trivial factorization $n = ab$. Then we also have a protocol which works for a kids. We just make b copies of each of them. By symmetry, we have a protocol which works for b kids. This is called the **Factor Rule**.

Suppose we have a protocol which works for a kids and a protocol which works for b kids. We do *not* necessarily have a protocol for ab kids *unless* the probability value for both protocols are the same. Then we divide the kids into a groups of size b, use the protocol for b kids to determine which group gets the coin, and then use the protocol for a kids within the lucky group. This is called the **Product Rule**.

The restriction in **Product Rule** vanishes when $a = b$, where a common probability value is guaranteed. Thus if we have a protocol which works for a kids, then we also have a protocol for a^b kids for any b. This is called the **Power Rule**.

Let's try to apply these rules. We have already seen in section 3 that flipping the coin four times and setting $r = \sqrt{\frac{1}{2\sqrt{3}} - \frac{1}{4}}$ works for three kids.

Suppose we toss a crooked coin five times. Then

$$\begin{aligned} 1 \quad = \quad & \left(\frac{1}{2}-r\right)^5 + \left(\frac{1}{2}+r\right)^5 + 5\left(\left(\frac{1}{2}-r\right)^4\left(\frac{1}{2}+r\right)\right. \\ & \left. + 2\left(\frac{1}{2}-r\right)^3\left(\frac{1}{2}+r\right)^2 + 2\left(\frac{1}{2}-r\right)^2\left(\frac{1}{2}+r\right)^3 + \left(\frac{1}{2}-r\right)\left(\frac{1}{2}+r\right)^4\right). \end{aligned}$$

Note that $(\frac{1}{2} - r)^5 + (\frac{1}{2} + r)^5 = \frac{1}{16} + \frac{5}{2}r^2 + 5r^4$. Setting this equal to $\frac{1}{6}$, $r = \sqrt{\frac{1}{2\sqrt{3}} - \frac{1}{4}}$ works for six kids. This is exactly the same value as the one obtained in the preceding case for three kids.

The **Product Rule** now yields a protocol for 18 kids. However, such a protocol can be derived from just the protocol for 6 kids, via a protocol for 36 kids. We apply the **Power Rule** followed by the **Factor Rule**.

Let's start by tossing the coin some number of times then look for an n that works. Let's start by tossing the crooked coins seven times. Then

$$\begin{aligned} 1 \quad = \quad & \left(\frac{1}{2}-r\right)^7 + \left(\frac{1}{2}-r\right)^7 \\ & +7\left(\left(\frac{1}{2}-r\right)^6\left(\frac{1}{2}+r\right) + 3\left(\frac{1}{2}-r\right)^5\left(\frac{1}{2}-r\right)^2 + 5\left(\frac{1}{2}-r\right)^4\left(\frac{1}{2}-r\right)^3\right. \\ & \left. +5\left(\frac{1}{2}-r\right)^3\left(\frac{1}{2}-r\right)^4 + 3\left(\frac{1}{2}-r\right)^2\left(\frac{1}{2}-r\right)^5 + \left(\frac{1}{2}-r\right)\left(\frac{1}{2}-r\right)^6\right). \end{aligned}$$

Note that $(\frac{1}{2} - r)^7 + (\frac{1}{2} + r)^7 = \frac{1}{64} + \frac{21}{16}r^2 + \frac{35}{4}r^4 + 7r^6$. Setting this equal to $\frac{1}{8}$, r is the unique positive root of $64r^6 + 80r^4 + 12r^2 - 1 = 0$. This works for eight kids.

Suppose we toss a crooked coins eight times. Then

$$
\begin{aligned}
1 \quad = \quad & \left(\frac{1}{2}+r\right)^8 + 4\left(\frac{1}{2}+r\right)^6\left(\frac{1}{2}-r\right)^2 + 6\left(\frac{1}{2}+r\right)^4\left(\frac{1}{2}-r\right)^4 + 4\left(\frac{1}{2}+r\right)^2\left(\frac{1}{2}-r\right)^6 + \left(\frac{1}{2}-r\right)^8 \\
& + 8\Bigg(\left(\frac{1}{2}+r\right)^7\left(\frac{1}{2}-r\right) + 3\left(\frac{1}{2}+r\right)^6\left(\frac{1}{2}-r\right)^2 + 7\left(\frac{1}{2}+r\right)^5\left(\frac{1}{2}-r\right)^3 + 8\left(\frac{1}{2}+r\right)^4\left(\frac{1}{2}-r\right)^4 \\
& + 7\left(\frac{1}{2}+r\right)^3\left(\frac{1}{2}-r\right)^5 + 3\left(\frac{1}{2}+r\right)^2\left(\frac{1}{2}-r\right)^6 + \left(\frac{1}{2}+r\right)\left(\frac{1}{2}-r\right)^7\Bigg).
\end{aligned}
$$

Note that $((\frac{1}{2}-r)^2+(\frac{1}{2}+r)^2)^4 = (\frac{1}{2}+2r^2)^4$. Setting it equal to $\frac{1}{3}$, we have $\frac{1}{2}+2r^2 = \frac{1}{\sqrt[4]{3}}$. Hence $r = \sqrt{\frac{2-\sqrt[4]{3}}{4\sqrt[4]{3}}}$ works for three kids. Setting it equal to $\frac{1}{5}$, we have $\frac{1}{2}+2r^2 = \frac{1}{\sqrt[4]{5}}$. Hence $r = \sqrt{\frac{2-\sqrt[4]{5}}{4\sqrt[4]{5}}}$ works for five kids. Setting it equal to $\frac{1}{9}$, we have $\frac{1}{2}+2r^2 = \frac{1}{\sqrt{3}}$. Hence $r = \sqrt{\frac{2\sqrt{3}-3}{12}}$ works for nine kids.

Suppose we toss a crooked coins nine times. Then

$$
\begin{aligned}
1 \quad = \quad & \left(\frac{1}{2}-r\right)^9 + 3\left(\frac{1}{2}-r\right)^6\left(\frac{1}{2}+r\right)^3 + 3\left(\frac{1}{2}-r\right)^3\left(\frac{1}{2}+r\right)^6 + \left(\frac{1}{2}+r\right)^9 \\
& + 9\Bigg(\left(\frac{1}{2}+r\right)^8\left(\frac{1}{2}-r\right) + 4\left(\frac{1}{2}+r\right)^7\left(\frac{1}{2}-r\right)^2 + 9\left(\frac{1}{2}+r\right)^6\left(\frac{1}{2}-r\right)^3 + 14\left(\frac{1}{2}+r\right)^5\left(\frac{1}{2}-r\right)^4 \\
& + 14\left(\frac{1}{2}+r\right)^4\left(\frac{1}{2}-r\right)^5 + 9\left(\frac{1}{2}+r\right)^3\left(\frac{1}{2}-r\right)^6 + 4\left(\frac{1}{2}+r\right)^2\left(\frac{1}{2}-r\right)^7 + \left(\frac{1}{2}+r\right)\left(\frac{1}{2}-r\right)^8\Bigg).
\end{aligned}
$$

Note that $((\frac{1}{2}-r)^3+(\frac{1}{2}+r)^3)^3 = (\frac{1}{4}+3r^2)^3$. Setting this equal to $\frac{1}{4}$, $r = \sqrt{\frac{4-\sqrt[3]{4}}{12\sqrt[3]{4}}}$ works for four kids as we saw in section 2. Setting this equal to $\frac{1}{10}$, $r = \sqrt{\frac{4-\sqrt[3]{10}}{12\sqrt[3]{10}}}$ works for ten kids.

We do not have a protocol which works for seven kids. Perhaps the reader can construct one. The inspiration for this problem comes from the Hungarian Mathematical Olympiad, called the Kurschak Competition.