From Untouchable 11 to Hazmat Cargo

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Untouchable 11 is a packing puzzle designed by Peter Grabarchuk. This paper describes Untouchable 11 and its 'untouchable' concept, and explores applying this concept to other hexomino packing puzzles. Every untouchable packing puzzle can be mapped to an equivalent conventional packing puzzle (in which pieces can touch), enabling the use of existing software tools for analysis. Exploring this puzzle space led to the creation of a new puzzle, Hazmat Cargo.

1 Introduction

UNTOUCHABLE 11 is a packing puzzle consisting of eleven pieces based on the eleven possible unfoldings of a cube, which themselves are a subset of the 35 *hexominoes*.[1] The goal is to place all eleven pieces onto a board such that no pieces touch, even diagonally at corners. The pieces can be rotated and flipped, but must be placed orthogonally onto the grid of the board. The puzzle offers three challenges:

1. *Easy* (9×17 board).
2. *Medium* (10×15 board, Figure 1).
3. *Hard* (12×12 board).

This paper describes how this idea of 'untouchable' packings has spread to other puzzles, and ultimately led to a new design of mine, described in a later section.

1.1 History

Untouchable 11, designed by Peter Grabarchuk,[2] first appeared on the gaming website SmartKit.com,[3] which sponsored the development of the associated app. In October 2008, it was launched with a contest[4] which gave a Smartkit t-shirt and the book *Puzzles' Express 3* [1] to the first person to solve all three challenges.

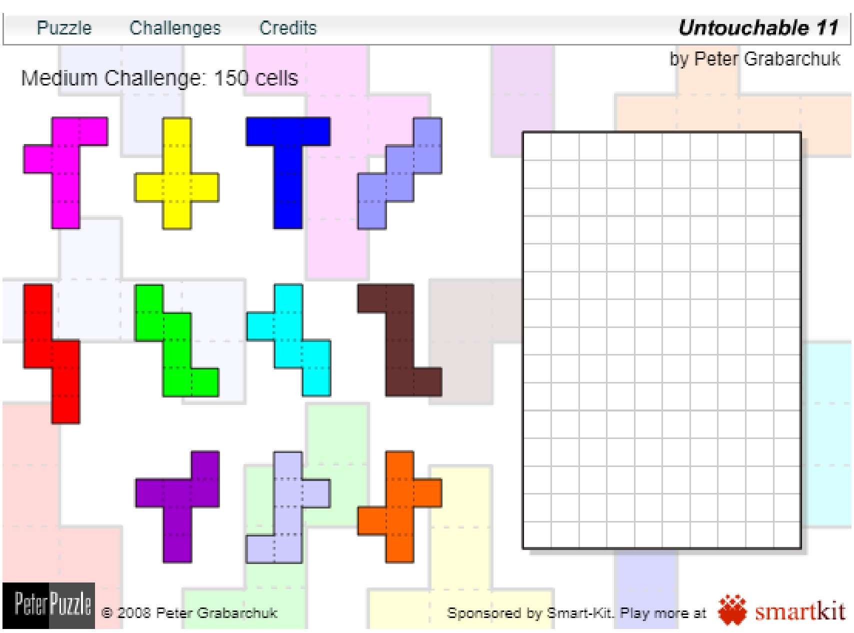

Figure 1. Screenshot of the medium (10×15) Untouchable 11 challenge.

[1] http://mathworld.wolfram.com/Polyomino.html
[2] http://www.grabarchukpuzzles.com
[3] http://smart-kit.com
[4] http://smart-kit.com/s1512

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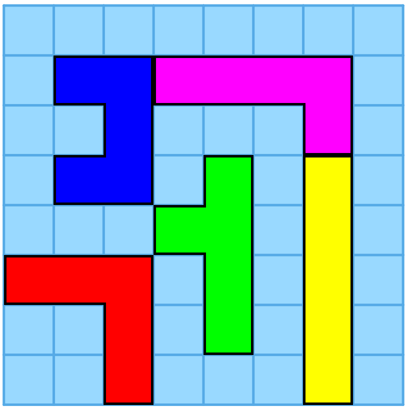

Figure 2. Solution to Golomb's problem.

The concept of a polyomino packing puzzle, in which no two pieces can touch even at a corner, appears to be original to the Grabarchuk family. In his book *Polyominoes* [2], Solomon Golomb asks what is the minimum number of pentominoes that can be placed on an 8×8 checkerboard such that none of the remaining ones can be added. The answer is five, and Figure 2 shows one such configuration. This sparse covering of the board seems to be a precursor to Grabarchuk's untouchable concept.

Kadon Enterprises, Inc.[5] also has a few games using similar concepts. Squint, a logic game played on a 9×12 grid, using their Quintillions set (1980). The goal is to make the last move by leaving no space on the grid for the opponent to place another quint (their brand name for pentomino).

Players in turn select a quint from the common pool and place it on the grid. The first quint must cover one of the board's corner squares. Later quints must be placed so that at least one of their corner points touches a corner point of any of the quints already on the board, and no sides may touch. Figure 3 shows such an arrangement.

This rule that corners must touch and sides may not touch results in a similarly sparse covering of the board. It also appears in the well-known game Blokus (2000) as a restriction on each player's own pieces.

Cornered is a similar logic game played using the Sextillions set. In that game, the pieces (the 35 hexominoes plus one duplicate) are divided between two players. In turn, players select one of their own pieces and place it on a 15×15 grid. A player's own pieces may not touch each other, not even diagonally at corners. A piece may touch opponent's pieces only at corners (no sides), but are not required to touch. The last player to put a piece on the board wins.

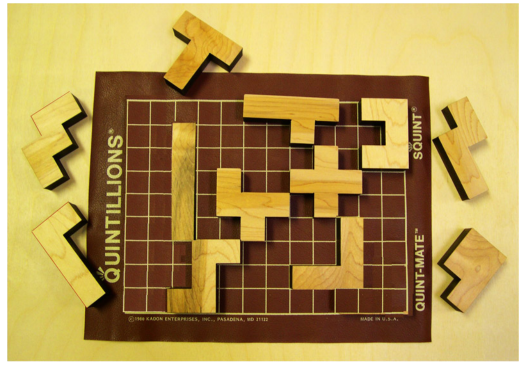

Figure 3. Squint example.

The only other puzzle I am aware of which uses the eleven unfoldings of a cube is a puzzle Kate Jones presented as her exchange gift at the 11^{th} Gathering for Gardner. She named this puzzle 11 Magic Cubes.[6] Other than using the same pieces, it bears little resemblance to Untouchable 11.

2 Solving

In 2008, I solved the easy and medium challenges by hand. After days of struggling with the hard challenge, the closest I came to solving it is shown in Figure 4.

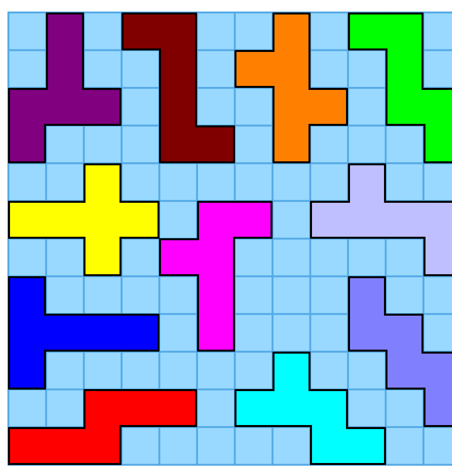

Figure 4. Near-solution to the hard challenge.

At this point, Peter was contacted and asked if the solution was unique. It turned out that the initial challenges were solved by Grabarchuk family members without the aid of computer algorithms. Peter knew of only two solutions to the hard challenge, and the total number of solutions was an unknown at that time. So now there were two puzzles to solve: I still needed to solve the

[5]http://www.gamepuzzles.com
[6]http://www.gamepuzzles.com/g4g11cubes.pdf

hard challenge, and — more interestingly — to count the total number of solutions!

Unable to find a solver capable of solving these untouchable packing problems, I created my own, shown in Figure 5. Algorithms for solving packing puzzles typically use a recursive backtracking search [3]. Knuth describes how to efficiently implement this type of search in his paper 'Dancing Links' [4]. Matt Busche also has an article[7] suggesting how to combine a number of relevant strategies and ideas, including those developed by de Bruijn [5] and Fletcher [6].

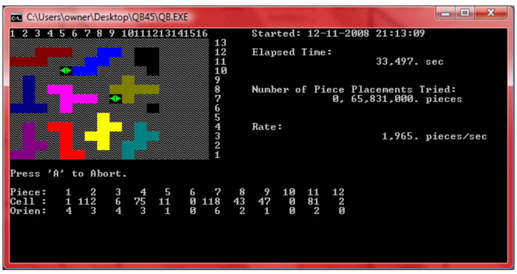

Figure 5. The author's Untouchable 11 solver.

My Untouchable 11 solver uses several of these strategies. The source code is in Quick Basic 4.5 and is available.[8] The code works and found all seven solutions to the hard challenge of Untouchable 11, but it took over 24 days to complete its search. The output of that initial search is available,[9] but be warned that it contains solutions.

However, before the 24-day search was completed, it became apparent that the puzzle could be mapped to a conventional (touching) packing puzzle. This would allow the use of many other existing solvers which are much more efficient.

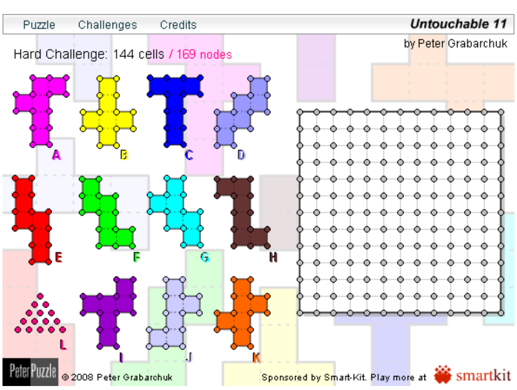

Figure 6. Mapping to a touching packing puzzle.

The idea is to map each original piece to a new piece defined by squares centred at vertices of the original piece, and increasing the width and height of the playing grid by one square. Figure 6 shows the original 12×12 challenge viewed this way: an equivalent task is to place the vertices onto the 13×13 grid of vertices. This results in exactly fifteen empty vertices.

In effect, this thickens each piece by wrapping it in an additional half-square wide layer. This additional part of each piece neatly fits into the required gaps between pieces in the original version of the puzzle. Each resulting piece is one square higher and one square wider. Figure 7 shows how two original pieces become two touching thicker pieces under this mapping.

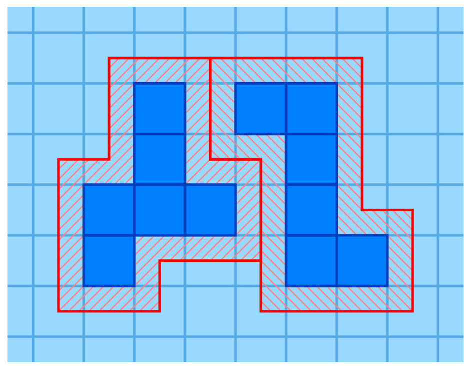

Figure 7. Half-unit thickening of pieces.

The fastest of the polyomino solvers that were readily available in 2008 was Gerard Putter's Polyomino Solver.[10] Once the hard challenge was mapped to its conventional touching equivalent and fed into this solver, the seven solutions were all found in under an hour. This work was completed before my 24-day search finished running.

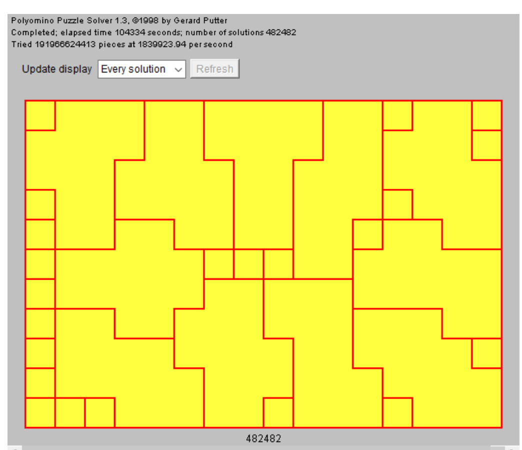

Figure 8. Result from Gerard's Polyomino Solver.

[7] http://www.mattbusche.org/blog/article/polycube
[8] http://wwwmwww.com/gapd/Untouch.TXT
[9] http://wwwmwww.com/gapd/SOLUTION_Finished.TXT
[10] https://gp.home.xs4all.nl/PolyominoSolver/downloadsolver.htm

The latter results confirmed the count and solutions found with Gerard's solver. Figure 8 shows output from Gerard's solver for the medium challenge. (We will not spoil the solution to the hard challenge here!) It found 482,482 solutions in 104,334 seconds (roughly 29 hours).

3 New Challenges

With a general solver, the first space to explore was additional rectangular boards as new challenges for these eleven original pieces. Table 1 shows these results. The 'Empty' column gives the number of empty cells in the mapped version, i.e. number of untouched vertices in the original version.

Board	Solutions	Name	Empty
12×12	7	Hard	15
11×13	33		14
10×15	482,482	Medium	22
9×16	174		16
9×17	65,516,235	Easy	26
8×18	15		17
7×21	60,327		22
6×24	8		21

Table 1. Solution counts for Untouchable 11 challenges.

Five new challenges were found that all fall between the medium and hard challenges in terms of difficulty. It was also proven that one entire row of the easy challenge, the 9×17 board, could be left empty, because the 9×16 board is solvable. Untouchable 11 now consisted of eight total challenges and received the Gamepuzzles Annual Polyomino Excellence Award for 2015.[11] Figure 9 shows the trophy.

A physical version of Untouchable 11 was created as the author's exchange puzzle for the 2017 *International Puzzle Party* (IPP37) in Paris, France. This puzzle included all eight challenges. The pieces were made of laser-cut acrylic by Sculpteo.[12] The board was 3D printed in Polyamide using selective laser sintering, SLS, by i.Materialise.[13] Figure 10 shows the final product.

Figure 10 does not show a solution, as two pieces touch at corners. A state with a single corner touch is known as a *near-solution*. These were counted for the original Untouchable 11 hard challenge in November, 2016, and 3,092 near solutions were found. This count was later verified by Landon Kryger in December 2016.

Figure 9. Gamepuzzles Annual Polyomino Excellence Award for 2015.

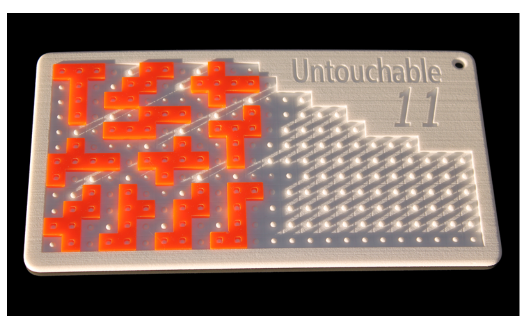

Figure 10. Carl Hoff's IPP37 exchange puzzle.

4 Widening the Search

The search for a set of eleven hexominoes which can be placed on a 12×12 board with a single unique solution was started in 2012. That work was done by creating modified code for each subset and running it through Gerard Putter's Polyomino Solver.

[11]http://www.gamepuzzles.com/gape15.htm
[12]https://www.sculpteo.com
[13]https://i.materialise.com

As each subset had to be coded by hand, this was slow tedious work, and the work was put on hold when a set with just two solutions was found. That set uses one hexomino which is not an unfolding of the cube. It was shared with Peter Grabarchuk and resulted in the release of Untouchable 11: Master Challenge[14] in March 2012, shown in Figure 11. This work was initially prompted by the need for an exchange gift[15] for the 10^{th} Gathering for Gardner, G4G10.

The search resumed late in 2016 with the assistance of programmers Brandon Enright and Landon Kryger. Landon had created a new, efficient solver which could test all possible subsets of a given size from a master set on a given board, to find puzzles with unique solutions.

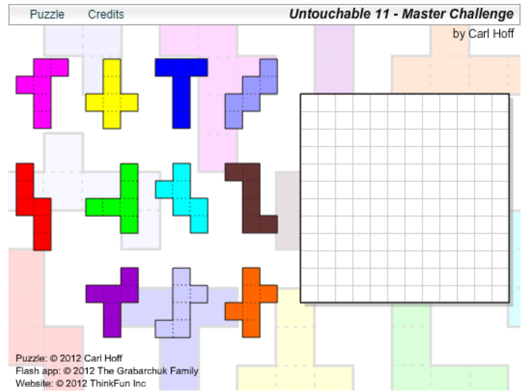

Figure 11. Untouchable 11: Master Challenge.

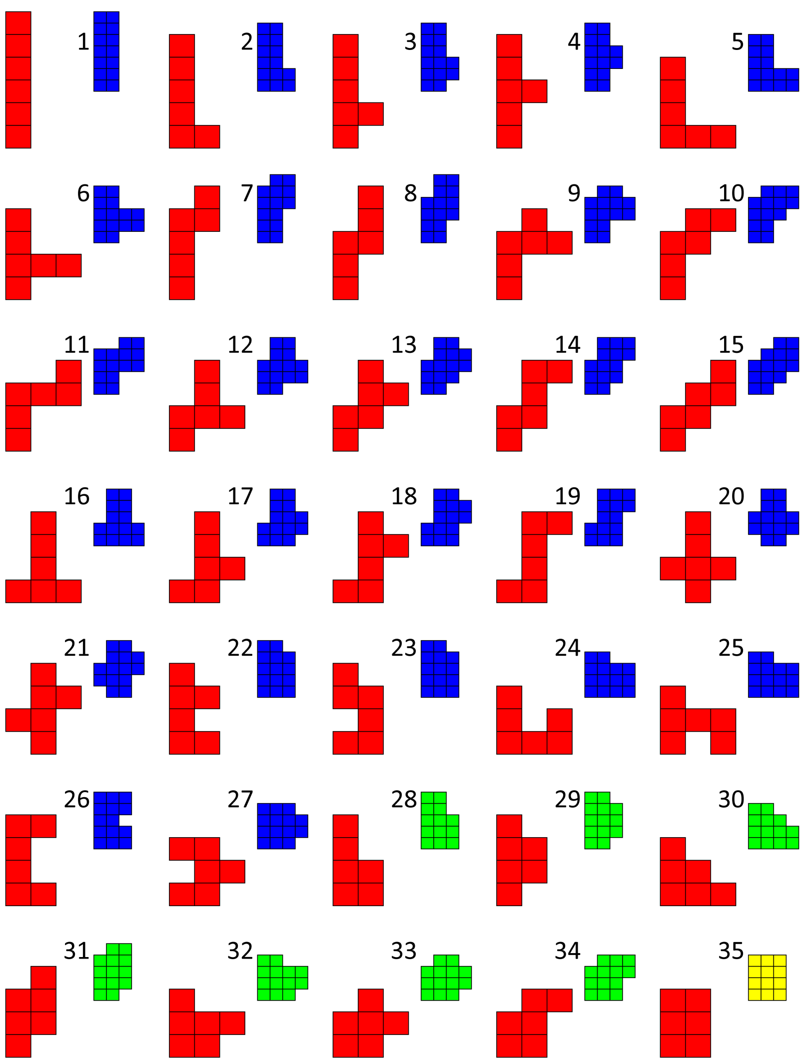

Figure 12. The 35 hexominoes and their vertex duals.

[14] http://www.puzzles.com/PuzzleClub/Untouchable11MasterChallenge
[15] http://wwwmwww.com/gapd/U11MasterChallenge.pdf

Board	N	Empty	Subsets	Tested	Search%	Single%	0 Solns	1 Soln	>1 Soln
5×5	2	8	210	210	100.0%	18.5714%	124	39	47
6×6	2	21	210	210	100.0%	0.0000%	0	0	210
6×6	3	7	1,330	1,330	100.0%	6.0902%	1,074	81	175
7×7	3	22	1,330	1,330	100.0%	0.0000%	0	0	1,330
7×7	4	8	5,985	5,985	100.0%	8.6717%	4,365	519	1,101
8×8	4	25	5,985	5,985	100.0%	0.0000%	0	0	5,985
8×8	5	11	20,349	20,349	100.0%	4.2017%	4,750	855	14,744
9×9	6	16	54,264	54,264	100.0%	0.0792%	199	43	54,022
9×9	7	2	116,280	116,280	100.0%	0.0439%	116,213	51	16
10×10	7	23	116,280	116,280	100.0%	0.0000%	0	0	116,280
10×10	8	9	203,490	203,490	100.0%	5.0980%	79,601	10,374	113,515
11×11	9	18	293,930	293,930	100.0%	0.0003%	0	1	293,929
11×11	10	4	352,716	107,010	30.3%	1.9325%	100,236	2,068	4,706
12×12	11	15	352,716	352,716	100.0%	0.0020%	49	7	352,660
12×12	12	1	293,930	293,930	100.0%	0.0024%	293,920	7	3
13×13	14	14	116,280	116,280	100.0%	0.0000%	116,280	0	0
14×14	16	0	20,349	20,349	100.0%	0.0000%	20,348	0	1

Table 2. Summary of search results. **N** indicates number of pieces.

The first thing to decide on was the master set that would be used: as shall be shown, there is no reason to include all 35 hexominoes, and a smaller set of candidates would mean a shorter search time. Figure 12 shows the complete set of 35 hexominoes and their vertex duals, created by mapping each vertex to a square, i.e. the thicker versions of each piece. 27 vertex duals have fourteen squares (shown in blue), but seven have thirteen squares (shown in green), and one has only twelve (yellow). We decided to use only the first 21 hexominoes as the master set. The hexominoes 22 through 35 were removed from consideration for the following reasons:

Hexominoes 28-35 have fewer than fourteen squares in their dual versions, so they seem easier to place. Hexominoes 22-35 can all be contained in a 3×3 or a 4×2 box. These are all more compact than the original eleven unfoldings of a cube, so they seem easier to place.

Hexominoes 22 and 23 both map to the same vertex dual polyomino. Any set containing both could never have a single solution, since those two pieces could always swap positions, so at least one must be excluded. Hexominoes 24 and 25 also both map to the same vertex dual polyomino. Hexomino 26 is unsuitable, as no vertex dual has a protruding square which could fit in the small gap on its right side. Therefore, any solution containing this piece produces a second solution with this piece rotated 180°.

After we selected the master set, the results shown in Table 2 were generated after many months of CPU time. We found seven sets of eleven hexominoes with unique solutions on the 12×12 board. Also note that there are seven sets of twelve hexominoes which also have unique solutions on the 12×12 board.

Table 3 shows all 11-piece and 12-piece sets with unique solutions. These are excellent puzzles left for the reader to solve. It may seem counter-intuitive, but the 12-piece sets are much easier to solve than the 11-piece sets. This is due to the availability of only a single empty node, which allows one to backtrack much sooner, thus simplifying the search.

Set	Hexominoes											
A	9	10	12	13	14	15	16	17	18	20	21	
B	8	9	10	11	13	15	17	18	19	20	21	
C	8	9	10	11	12	13	15	17	18	20	21	
D	8	9	10	11	13	15	16	17	18	20	21	
E	1	8	9	11	12	13	15	17	18	20	21	
F	1	11	12	13	14	15	16	17	19	20	21	
G	4	8	9	12	13	15	16	17	18	20	21	
a	1	2	3	4	5	6	7	8	10	11	13	14
b	1	2	3	4	5	6	7	8	10	11	13	18
c	1	2	3	4	5	6	9	11	12	13	18	21
d	2	3	4	5	6	7	9	10	11	13	16	18
e	1	2	4	5	6	8	9	10	11	12	14	18
f	1	2	3	4	5	7	8	10	11	12	14	16
g	1	2	3	5	6	7	8	9	11	14	16	19

Table 3. 11 (A–G) and 12 (a–g) piece hexomino sets.

During this search, the need arose to design a puzzle for the IPP37 design competition. The initial candidates were the 11-piece sets with unique solutions seen above. But it was thought that these may be too difficult to be fully appreciated by most puzzlers, so smaller boards and fewer pieces were also considered. The search was focussed on square boards, because a compact board (with a lower perimeter-to-area ratio) will typically maximise difficulty. A good example of this is the Eternity Puzzle,[16] whose nearly circular board is shown in Figure 13.[17]

[16] http://www.mathpuzzle.com/eternity.html
[17] Figure derived from: http://www.archduke.org/eternity/solution/index.html

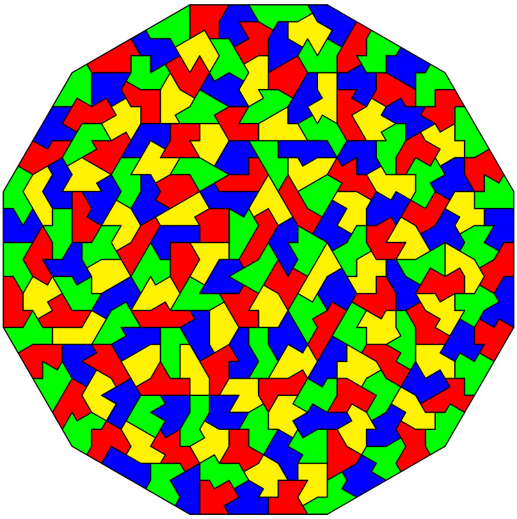

Figure 13. Eternity Puzzle solution by Alex Selby. Eternity pieces by Christopher Monckton © 1999.

Out of the 293,930 sets tested for the 9-piece puzzle on an 11×11 board, only a single set was found to have a unique solution. This set was explored and found to be very interesting for the following reasons:

1. It has only a single solution.
2. It uses a square board.
3. It has 3,761 near-solutions.

This single set has more near-solutions than the 3,092 near-solutions of the Untouchable 11 hard challenge, which has seven actual solutions. This puzzle also contained eighteen empty cells in its mapped version, three more than the original Untouchable 11 hard challenge. This is the largest number of empty cells found in any puzzle of this type with a unique solution to date.

The nine hexominoes found in this set are 8, 9, 12, 13, 15, 17, 18, 20, and 21. This puzzle was the one chosen for the design competition. In keeping with the untouchable theme, the pieces are physically designed to resemble groups of six industrial drums containing hazardous materials.

The board was designed to suggest a barge. The goal is to pack the nine groups of six hazmat drums onto the barge, an 11×11 array, such that no two pieces touch, not even at corners. (Any contact could lead to a catastrophic chemical reaction!) Figure 14 shows the puzzle submitted to the competition. All components were designed in SolidWorks,[18] and 3D printed in steel or polyamide by i.Materialise or Shapeways.[19]

[18]http://www.solidworks.com/
[19]https://www.shapeways.com/

5 Open Questions

Here are two open hypotheses, neither of which have been proven:

1. The 9-piece set used in Hazmat Cargo is the only 9-piece subset of the hexominoes to have a single solution on the 11×11 board.
2. All other 9-piece subsets have multiple solutions on the 11×11 board; there are none with no solutions.

There are $\binom{35}{9} = 70,607,460$ possible 9-piece subsets of the 35 hexominoes. Of these, only 293,930 have been searched, i.e. only about 0.42%. The sets that have been searched contain the hardest-to-place pieces.

Since they all have solutions, it is believed that adding easier-to-place pieces to the mix will not result in sets without solutions, or other sets with just a single solution. Still, neither hypothesis can be asserted with certainty. Please contact the author if you are able to prove either hypothesis.

There is also the question of what fun and interesting puzzles may exist in the space of untouchable hexomino packing puzzles with rectangular boards. That is the next task slated for Kryger's solver. If the piece sets are expanded to include other polyominoes and the board shapes are not restricted to just squares or rectangles, then there are even more possibilities.

6 Conclusion

While Hazmat Cargo did not win any awards at the design competition, it did receive numerous compliments, including the thematic barge and hazmat drums. Several commented that the physical design fit the untouchable concept perfectly. It was fun to design and took on a significantly different aesthetic than my previous designs.

Aside from the simple pleasure of designing a new puzzle, the lesson here is to take a new look at the puzzles you have enjoyed. In this case it was Peter Grabarchuk's Untouchable 11, which introduced a new concept to polyomino packing puzzles. This concept proved to open a very vast and interesting area which proved worthy of exploration. Five new challenges were added to the original Untouchable 11 puzzle. The Untouchable 11: Master Challenge was created and resulted in a new app being released and enjoyed. And the exploration resulted in a very difficult 9-piece puzzle named Hazmat Cargo.

Figure 14. The Hazmat Cargo puzzle.

Acknowledgements

Thanks to Peter Grabarchuk for his efforts in creating Untouchable 11, his permission to use that puzzle as my exchange puzzle at IPP37, and his blessings on writing this article. Thanks to Brandon Enright and Landon Kryger for all the assistance they have provided. Using Gerard Putter's Polyomino Solver alone, it would have taken me twenty years to search the 352,716 subsets needed for just the 11-piece 12×12 puzzle, while Kryger's solver reduced that time to a couple of months. Thanks to Gerard Putter for sharing his polyomino solver and Jaap Scherphuis for sharing his polyform puzzle solver[20], both of which were used in this study, with Jaap adding new functionality to his solver at my request. Thanks to Kate Jones for providing the information on Squint and Cornered. Please contact me for solutions to challenges presented in this paper.

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[20]https://www.jaapsch.net/puzzles/polysolver.htm