

(Phi)ve is a Magic Number

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Martin Gardner would surely have noted the centennial significance of January 6 of this year. On that day, the digits of the date 1/6/18 matched 1.618 – the leading digits of the famous golden ratio number Phi. If you wanted three more digits, you could have set your alarm to celebrate at 3:39 am or at a more reasonable 3:39 pm when the eight digits of that moment aligned to 1.6180339.

I am sure that Martin would have also noticed the puns in my title. The first word refers to both Phi and five, and the fourth word refers to magic squares. My goal is to reveal several surprising connections between the two values and their powers using both Fibonacci numbers and five-by-five magic squares. A warning: if ordinary word puns tend to make you numb, then my mathematical puns will make you number.

As you presumably know, the golden ratio number Phi (or ϕ) shows up in nature, in art, in architecture, and even beauty salons, as well as mathematics. If you are not familiar with ϕ , you are in for a treat. There are plenty of books and websites that explain the fascinating properties and ubiquitous nature of this mysterious number. I want to use a few of the many identities, so here is a short summary.

One of the ways to determine the golden ratio is to find the place to divide a line so that the ratio of the length of the longer segment to the shorter is the same as the ratio of the whole line to the longer segment. If x is the length of the longer segment and y is the length of the shorter, equating those two ratios gives the equation $\frac{y}{x} = \frac{y+x}{y}$ or $y^2 = xy + x^2$. We are only interested in the ratio, so we can arbitrarily set the value of the smaller value x to 1, to get a quadratic equation in a single variable, $y^2 = y + 1$, or $y^2 - y - 1 = 0$. Then you can find the two roots using the familiar quadratic formula, $\frac{-b \pm \sqrt{b^2 - 4ac}}{2}$. There is one positive root, namely, $\frac{1+\sqrt{5}}{2} = 1.6180339\dots$, which is the golden ratio, ϕ . The other root is $\frac{1-\sqrt{5}}{2} = -0.6180339\dots$, which can also be expressed as $1 - \phi$ or also as $\left(-\frac{1}{\phi}\right)$ or $(-\phi)^{-1}$.

The identities that I will use later are $\phi^2 = \phi + 1$ and $\phi^{-1} = \phi - 1$. Both of these are directly obtainable from the above. Also, instead of expressing ϕ as a fraction and a square root, we could use decimals and powers expressed as decimals, giving the interesting form:

$$\phi = 0.5 + 0.5(5)^{0.5}$$

It is not the easiest form to remember, but it does display the intimate connection between ϕ and 5. This is only the first of many links to be discovered.

Phi, Five, and Fibonacci

As you probably know, Fibonacci numbers are a sequence of integers defined by the rule that each number in the sequence is the sum of the previous two. If F_n denotes the n^{th} Fibonacci number, $F_n = F_{n-1} + F_{n-2}$. Along with that rule, you must know that the first two Fibonacci numbers are 1's, i.e., $F_1 = 1$ and $F_2 = 1$. So the sequence begins $\{1, 1, 2, 3, 5, 8, 13, \dots\}$.

If you are interested in how rapidly that sequence is increasing, you could examine the ratio of each Fibonacci number to the previous one, i.e., $\frac{F_n}{F_{n-1}}$. For example, $\frac{F_2}{F_1} = 1$, $\frac{F_3}{F_2} = 2$, $\frac{F_4}{F_3} = 1.5$, and so forth. Most people who are familiar with both ϕ and the Fibonacci numbers are aware the limiting value of those ratios is ϕ . That is, $\lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} = \phi$. Fewer people seem to be aware that there is a formula that expresses all of the Fibonacci numbers *exactly* in terms of ϕ . It is known as Binet's formula, published in 1843, although it was actually found and published much earlier by both Euler in 1756 and de Moivre in 1730.¹ Here it is:

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

This is an interesting result because the square root of 5 and all the non-zero powers of ϕ are irrational, yet the parts combine to produce exact integers for every value of n . Although we normally think of the Fibonacci sequence as containing only positive integers, the same formula works for zero and negative n , so the sequence can extend 'backwards.' That is, $F_0 = 0$, $F_{-1} = -1$, $F_{-2} = -1$, $F_{-3} = -2$, and so forth. As you can see, the negative sequence just mirrors the ordinary positive one.

Putting this formula into a form that emphasizes ϕ 's and 5's, the result can be expressed as:

$$F_n = 5^{-0.5}(\phi^n - (-\phi)^{-n})$$

The 5th Fibonacci number happens to be 5, so 5 and ϕ are linked by this surprising equation.

$$5 = 5^{-0.5}(\phi^5 - (-\phi)^{-5})$$

By the way, the seventh Fibonacci number is 13, so in honor of the thirteenth gathering,

$$13 = 5^{-0.5}(\phi^7 - (-\phi)^{-7})$$

Generalized Fibonacci sequences use the same recursion rule (the sum of the previous two) but with arbitrary starting values. If the n^{th} number in the generalized Fibonacci sequence is G_n , and the starting values are $G_0 = x$ and $G_1 = y$ for any x and y , then

¹ Deriving this formula using standard methods for solving a second order difference equation with two initial conditions is an easy and very satisfying exercise, suitable for high school algebra students.

$$\begin{aligned}
G_2 &= x + y \\
G_3 &= x + 2y \\
G_4 &= 2x + 3y \\
G_5 &= 3x + 5y
\end{aligned}$$

and so on. You can see from the way the coefficients of x and y are developing that the generalized Fibonacci numbers are closely related to the ordinary Fibonacci numbers by the equation $G_n = F_{n-1}x + F_n y$. The formula works for negative n and even for non-integer x and y . All of these results are well known, along with many other interesting features of the Fibonacci numbers.

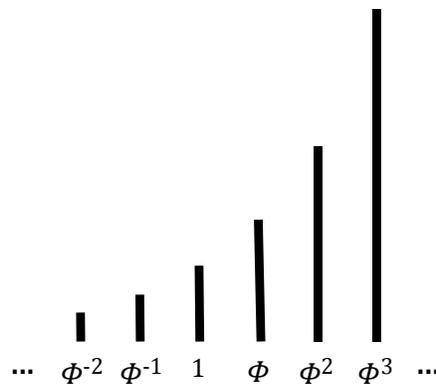
One particular instance of the generalized Fibonacci sequence that is relevant in the present context uses the initial values 1 and ϕ . I will call that sequence of numbers the Phi-bonacci sequence² and designate the terms as ϕ_n . So,

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= \phi \\
\phi_2 &= 1 + \phi \\
\phi_3 &= 1 + 2\phi \\
\phi_4 &= 2 + 3\phi
\end{aligned}$$

and so forth. In general, for all n , $\phi_n = F_{n-2} + F_{n-1}\phi$. The reason that sequence is interesting and relevant is that the closed form solution to the difference equation is very simple. It is:

$$\phi_n = \phi^n$$

That is, the n^{th} number in the sequence is the n^{th} power of ϕ . The same formula works for negative n , so $\phi_{-n} = \phi^{-n}$. When you think of ϕ as the golden ratio representing the ideal proportioning of line segments, the Phi-bonacci sequence is a set of line segments extending to infinity in both directions in which each one is multiplied by ϕ to match the larger neighbor and divided by ϕ to match the smaller neighbor.



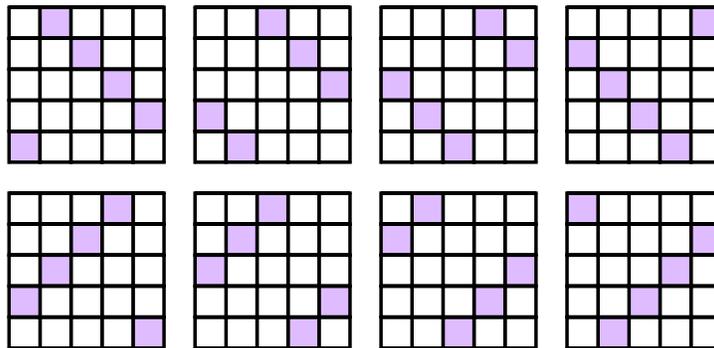
² Others have used this term for different sequences, but I think the use in this context is clear.

Phi, Five, and Magic Squares

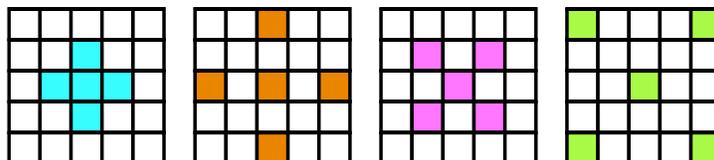
The following is a magic square of order five whose magic sum is 1.6180. Every row, column, and diagonal adds to the first five significant digits of ϕ .

0.3224	0.3246	0.3243	0.3235	0.3232
0.3238	0.3230	0.3227	0.3244	0.3241
0.3247	0.3239	0.3236	0.3233	0.3225
0.3231	0.3228	0.3245	0.3242	0.3234
0.3240	0.3237	0.3229	0.3226	0.3248

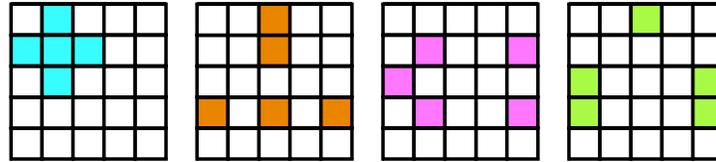
However, it is not just an ordinary magic square. It consists of the twenty-five consecutive decimal numbers starting with 0.3224 in the upper left cell and ending with 0.3248 in the lower right cell. It is pandiagonal, which means that all eight of the broken diagonals also sum to the same value.



You can also shift the rows or columns in any direction, wrapping around as you do so, and the square will remain magic. In addition to the twenty 'magic' patterns that you get from the five rows, five columns, five left diagonals, and five right diagonals, there are four that involve the center cell and four symmetrically surrounding cells.

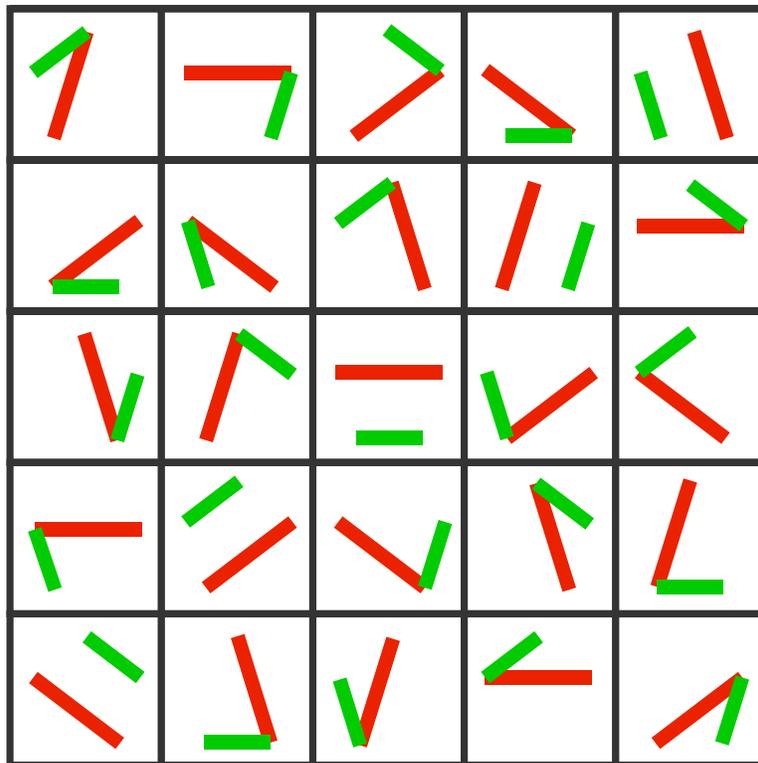


Each of these can be shifted so that center is in any of the 25 cells (wrapping around as necessary) so each of these four patterns has 25 variations, for a total of 120 patterns whose cell values add to the magic sum of 1.6180. For example, the values in the following patterns match the magic sum.

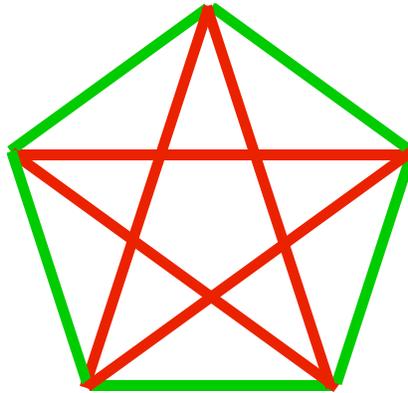


Furthermore, the magic square is symmetric, which means that any two cells that are at an equal distance from the center on a straight line through the center will contain values that sum to 40% of the magic sum, or 0.6472. That implies that any such opposite pair, together with any other such opposite pair and the center cell will match the magic sum. There are actually 780 combinations of five cells that match the magic sum! (You may use a calculator or a spreadsheet to check if you wish.)

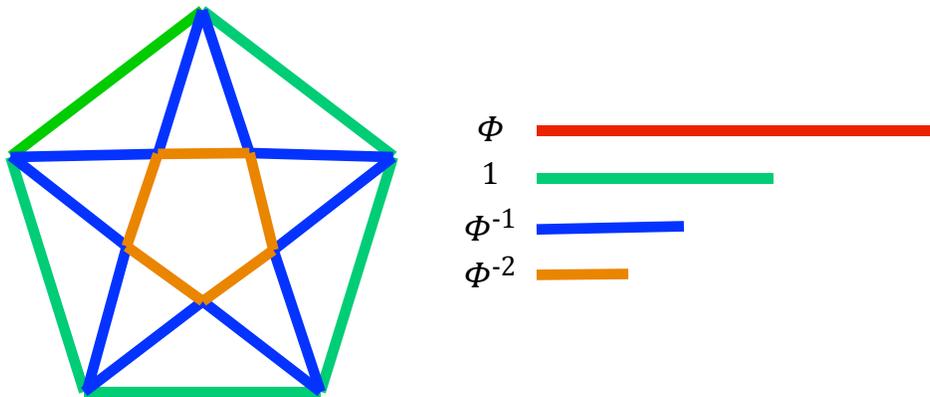
Of course, a purist would point out that this square does not really involve the true ϕ , but only a truncated approximation of it. In order to correct that flaw, I will resort to geometry. Each cell will contain two line segments: a red one of length ϕ and a green one of length 1 (in arbitrary units). When we add the geometric entries in cells, we simply superimpose them. With that geometric definition of addition, the following square is magic.



The square is pandiagonal, so all of the broken diagonals combine to the same result. The four patterns around the central cell, as well as the shifted versions also produce the pattern, so you get 120 ways that five cells form the magic sum! All 120 patterns sum to same figure, a five-pointed star (or pentagram) inscribed in a five-sided regular polygon (or pentagon). This symbol has been widely used throughout history to represent magical powers and as an instrument for casting magic spells, either evil or to protect against evil. More germane to our interest here, it contains many instances of ϕ and its powers.



We already know that every red line is of length ϕ . If each red line is broken into the three segments at the crossing points, every blue line segment is of length ϕ^{-1} and every orange segment is of length ϕ^{-2} .



Alternatively, if you want to measure all of the line segments relative to the shortest orange ones, taking those as the unit of measure, the blue lines would have length ϕ , the green ones would have length ϕ^2 , and the longest red lines would have length ϕ^3 . All of these powers of ϕ are successive numbers in the Phi-bonacci sequence.

The magic square is pandiagonal, so all of the broken diagonals combine to the same result. The four patterns around the central cell, as well as the shifted versions also produce the pattern, so you get 120 ways that five cells form the pentagram-in-pentagon figure.

If you take the numerical values of the line segments in the magic sum, they total $5 + 5\phi$, or using the identity $\phi^2 = \phi + 1$, the magic sum is $5\phi^2$. If you calculate what that amounts to, it is slightly more than 13, which seems fitting for the thirteenth G4G. If you add up the values of all of the line segments in the entire square, the total is $(5\phi)^2$, which seems appropriate for a five-by-five magic square designed to celebrate 5 and ϕ . It seems only fair to leave some fun for others, so I will only suggest that the perimeters and areas of the many internal polygons contain powers of ϕ and 5.

Finally, I would like to point out that Martin Gardner published his column in *Scientific American* starting in 1956 and ending in 1981—a total of 5^2 years. Can there be any doubt that (Phi)ve is a magic number?

References:

There are many books and articles about Phi, Fibonacci numbers, and magic squares. A quick search on the internet will produce enough to keep you entertained for weeks. If you want to learn how I created the magic squares (easily), the method is fully explained in my 2017 book, *More Magic Square Methods and Tricks*,