Hamming Code in a Magic Trick

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1 Error Detection and Correction Trick (Hamming Code)

Description: A volunteer thinks of a number and a color, the magician shows cards with several numbers for the volunteer to say whether he sees his number, but he can lie on one card, according to the chosen color. The magician is able to find on which color the volunteer lied, and then tell the number.

Material: Copy and cut cards on Appendix, if you have crayons you could color the cards accordingly.

Preparation: Put the cards in order, they are numbered. Practice how to check parity (see instructions below).

Performance: Gisele, the magician, will read Arthur's mind.

1. Gisele asks Arthur to pick a number between 1 and 15, and one of the colors of the rainbow (red, orange, yellow, green, blue, indigo or violet);

2. Gisele explains that Arthur has to say whether he can see the chosen number on each of the cards she shows;

3. But Gisele also explains that Arthur should tell a lie on the card having the color he chose;

4. For every time he says "yes" for a card, Gisele lays it facing-up, otherwise, if he says "no" she puts the card facing -down;

5. She arranges the cards side-by-side from left to right;

6. Once all seven cards are dealt, she looks at the cards and she can tell which color was chosen, and which number was selected.

Trick: Trick is based on the Hamming Code. The first four cards resemble a binary-digit trick with numbers 1 to 15. If there were no lie allowed, then we'd only need the first four cards. Simply, we would add the top left number on each card that faces up.

However, we are also trying to discover where the lie happened. This is similar to a computer system trying to fix a denigrated data communication. We need more digits (cards).

For every face-up card, you consider a 1 on the sequence. Face-down cards represent 0.

The first four cards will serve to compute the chosen number by adding the top-left number on cards facing up.

The last three cards are the "parity digits".

Check the parity digits (last three cards, see below), if only one of them does not match, then that is the "lie"-card, and the chosen number is the sum of the digits on top left on the first four cards that face up;

If more than one parity digit does not match, then you follow the rule, according to discrepancy:

- Parity 1 and 2: the person lied on the card 3 (yellow), so you flip the card, and the chosen number will be the sum of the numbers on top left on the first four cards who face up (after you "fixed" the sequence);

- Parity 1 and 3: the person lied on the card 2 (orange), so you flip the card, and the chosen number will be the sum of the numbers on top left on the first four cards who face up (after you "fixed" the sequence);

- Parity 2 and 3: the person lied on the card 1 (red), so you flip the card, and the chosen number will be the sum of the numbers on top left on the first four cards who face up (after you "fixed" the sequence);

- Parity 1, 2 and 3: the person lied on the card 4 (green), so you flip the card, and the chosen number will be the sum of the numbers on top left on the first four cards who face up (after you "fixed" the sequence).

Checking Parity Digits
Once you put the cards, consider face-up to be 1, face-down to be 0;

- The first parity digit (the fifth card/digit) needs to make even the sum on digit 2, 3, and 4. For instance, if only one of the digits 2, 3, and 4 is facing-up, then the first parity digit needs to also be 1 (facing-up);

- Second parity digit (sixth card/digit), checks the parity on cards 1, 3, and 4.

- Third parity digit (last card), checks the parity on cards 1, 2 and 4.

Explanation: You are creating a sequence of seven digits 0's and 1's. Because there is a lie, the final sequence will not be any one on the table on "Error Detection" section. With multiple parity checks, we are able to identify the incorrect digit, fix it, and find the correspondent number.

Hint: Practice the error recognition. At first, it may take you a while to figure out the lie. Fix the lie, before telling the chosen number.

Example 1: Suppose that the chosen number is 13, and the chosen color is indigo.

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
Yes	Yes	No	Yes	No	Yes (lie)	Yes
1	1	0	1	0	1	1

Parity 1 (card 5): since between cards 2, 3, and 4, there are two 1's, then it was supposed to be 0. Ok.

Parity 2 (card 6): since between cards 1, 3, and 4, there are two 1's, then it was supposed to be 0. Error.

Parity 3 (card 7): since between cards 1, 2, and 4, there are three 1's, then it was supposed to be 1. Ok.

Since, there is only one parity digit wrong, then that's where the lie is. The chosen number is 8+4+1=13.

Example 2: Suppose that the chosen number is 5, and the chosen color is orange.

Card 1	Card 2	Card 3	Card 4	Card 5	Card 6	Card 7
No	No (lie)	No	Yes	No	Yes	No
0	0	0	1	0	1	0

Parity 1: since between cards 2, 3, and 4, there is only one 1, then it was supposed to be 1. Error.

Parity 2: since between cards 1, 3, and 4, there is only one 1, then it was supposed to be 1. Ok.

Parity 3: since between cards 1, 2, and 4, there is only one 1, then it was supposed to be 1. Error.

Since, parity digits 1 and 3 show discrepancy, the lie is on the second card. Once we fix it, we calculate that the chosen number is 4+1=5.

Extra: How are the cards created?

Now you know how to perform the trick, but how did we place the numbers on the cards in figure 2 so that this trick works? The first four cards give the binary expansion of the chosen number; so these cards correspond to 8, 4, 2, and 1. A number appears on one of these cards if the associated power of two appears in the binary expansion for the number. For instance, the binary expansion for 13 is 13=8+4+1, so 13 appears on the first, second, and fourth cards.

The other three cards are the parity check cards. A number appears on the fifth (blue) card if its binary expansion has an odd number of the following powers of two: 4, 2, and 1. Hence, 9=8+1 and 7=4+2+1 are on the fifth card, but 5=4+1 is not.

The numbers on the sixth and seventh cards are determined similarly. Numbers on sixth card have an odd number of the following powers of 2: 8, 2, and 1. Numbers of fifth cards have an odd numbers of 8, 4, 1.

Further Reading

This magic trick appears on an article by Ricardo Teixeira on the February 2017 edition of Math Horizon, with the title of "Magical Data Restoration." It is a refinement of a trick created by Richard Ehrenborg ("Decoding the Hamming Code," Math Horizons, April 2006) and enhanced by Todd Mateer ("A Magic Trick Based

on the Hamming Code," Math Horizons, November 2013). Ehrenborg's and Mateer's tricks, while essentially the same as this one, require specially designed cards with a clever but little complicated system of tabs to aid the magician. The advantage and disadvantage of using simple rectangular cards is that the trick happens in your mind, instead of physically on the cards.

2 Appendix: Cards

| 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 |

1: Red Card

| 4 | 5 | 6 | 7 |
| 12 | 13 | 14 | 15 |

2: Orange Card

| 2 | 3 | 6 | 7 |
| 10 | 11 | 14 | 15 |

3: Yellow Card

| 01 | 3 | 5 | 7 |
| 9 | 11 | 13 | 15 |

4: Green Card

| 01 | 2 | 4 | 7 |
| 9 | 10 | 12 | 15 |

5: Blue Card

| 01 | 2 | 5 | 6 |
| 8 | 11 | 12 | 15 |

6: Indigo Card

| 01 | 3 | 4 | 6 |
| 8 | 10 | 13 | 15 |

7: Violet Card

| Parity 1 (card 5) | Parity 2 (card 6) | Parity 3 (card 7) |
| 2,3,4 | 1,3,4 | 1,2,4 |

Cheat Sheet