Abstract

This paper is to formally introduce tetraflexagons that are completely analogous to hexaflexagons as introduced by Martin Gardner in Scientific American over sixty years ago, and place them on the same mathematical platform as their six sided cousins.

In order to be a true flexagon, the thing must show all of its faces without backing up.

This is all based on original work since I discovered true tetraflexagons in 1961. Until recently, I believed mine were original never before published, but that is not so.

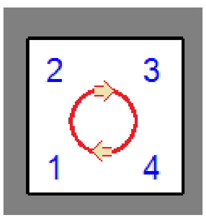
I am proposing a systematic and visual method to distinguish one flexagon from another with the same number of faces, and serve to count distinct varieties.

I am using flexagons to introduce my Penrose pattern coloring system as I can have several generations of the pattern on a single flexagon, or other artistic varieties. In the past, I've written letters on tetraflexagons as that avoids separate pages after front and back and they fold in half to a letter shaped thing.

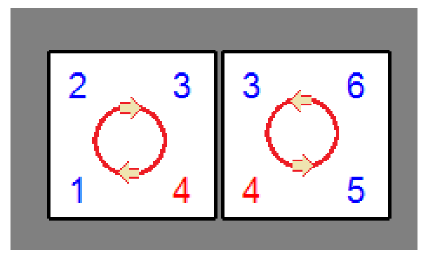
Tetraflexagons

All flexagons can be classified by the sequence in which they show their faces. Tetraflexagons have four sides and show their faces in cycles of four.

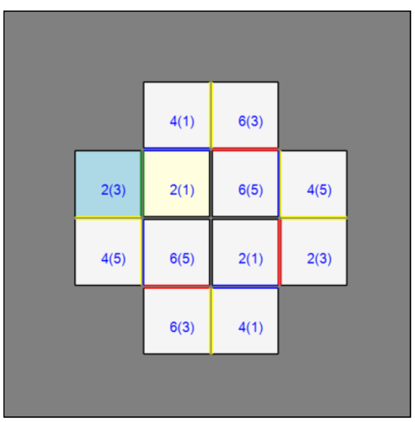
These two flex cycle diagrams are for the primordial pair of tetraflexagons.

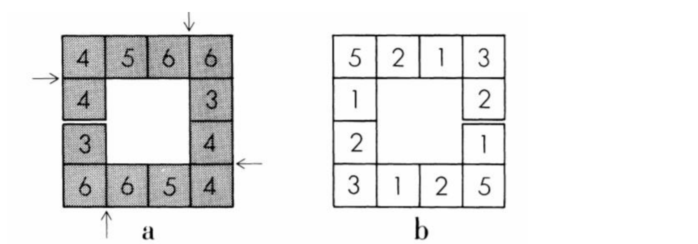


I call these drawings "Signatures". The direction of the cycles is indicated by the arrows and the red numbers are the faces where you have a choice to stay on the current cycle, or switch to the adjacent one.



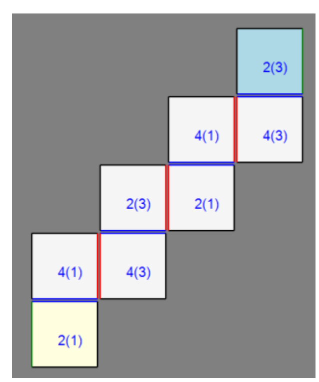
In 1961 I read of tetraflexagons, but I was not able to correctly fold the example from the book*. With scissors and tape I was able make it work like a hexaflexagon. The paper design looks like the pattern on the left.





Hexa-Tetraflexagon map. "Square of squares"

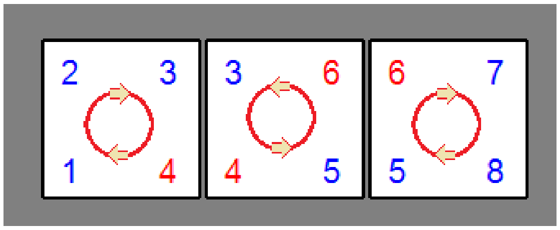
This map is the correct shape for the hexatetraflexagon. The primordial tetratetraflexagon is folded from a shape like one on the right. I found this at the same time.



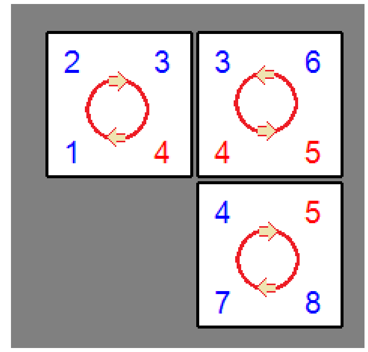
It did not take long to see how to add additional faces to working flexagons after "fixing" the square of squares.

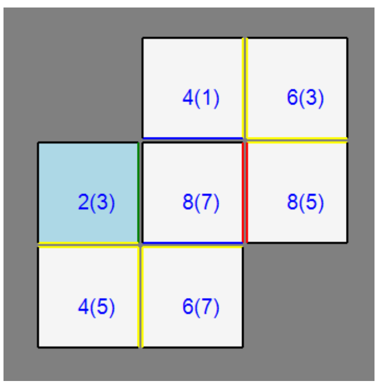
It turns out there are two distinct varieties of octatetraflexagons. I number the faces in the order in which they are added to the signature, but keep all the even numbers on the front and odd numbers on the back.   New faces are always added in pairs as no true tetraflexagon can have an odd number of faces.

**By adapting the Internet method for the flat square of squares to the version with the cut and altering the folding instructions slightly, it worked without the glitch from the seamless square of squares.   This sort of changes everything.*

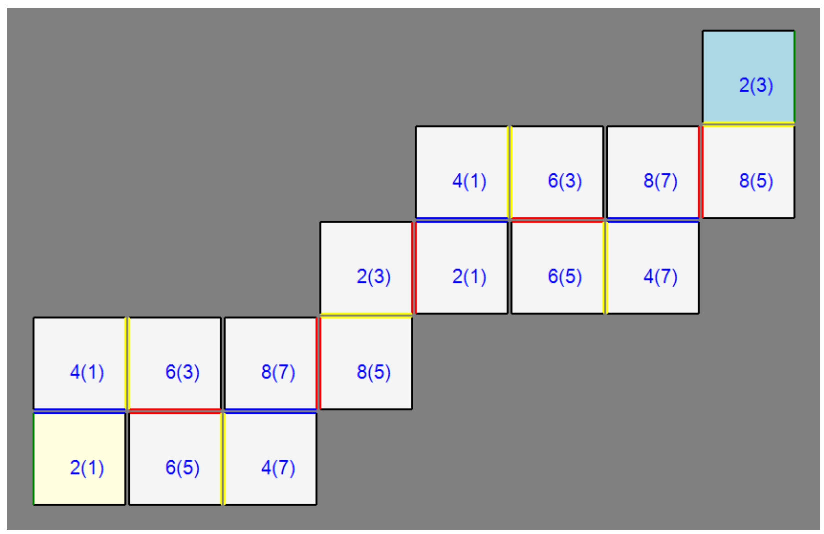


The signature on the left can traverse all eight faces in sequence, whereas the one on the right cannot.   They are made from very distinctly different maps.   In Martin Gardner's book, flexagons that are able to show all of their faces in sequence were called "street" flexagons, and their signatures are also always straight.   Thus there is always exactly one flexagon with a given number of faces that is capable of exhibiting this characteristic.



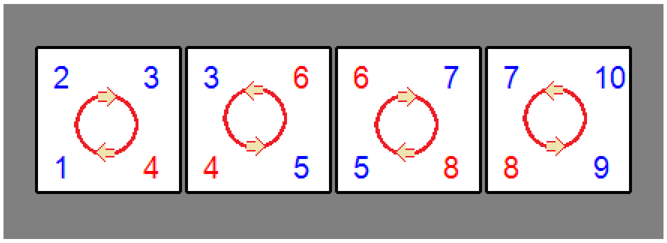


The map for the "street" octatetraflexagon also shares a characteristic with the seven faced street hexaflexagon in Martin Gardner's book: they both have "spiral" maps which when laid flat, have multiple layers.   The above map has four layers.

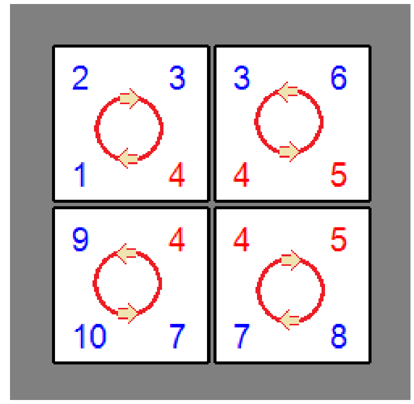


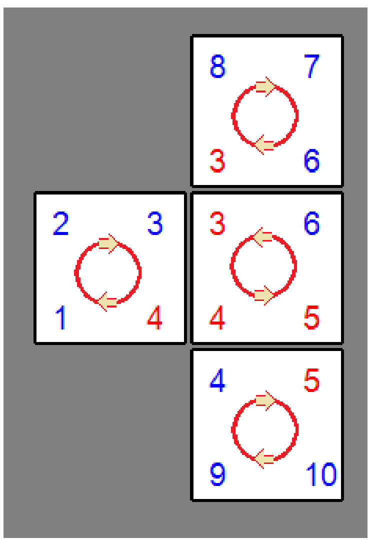
In these maps, I also show the joints between the squares – a blue joint is fold-back from the center and red is fold up from the center, and yellow is fold both ways and be able to have other squares sit between them when folded. Green is for the "glue" joint that has to be made after the flexagon is folded and it will meet with the glue joint at the other end of the strip.   By folding the squares together in reverse face order, two at a time, the partially folded strip will resemble the strip of the flexagon from which it was made. This is also true for hexaflexagons. The numbers in ()s are the face numbers on the back.

Carrying on from octatetraflexagons, it turns out there are five distinct varieties of decatetraflexagons.

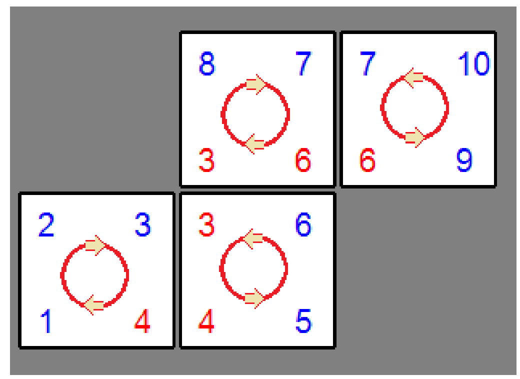


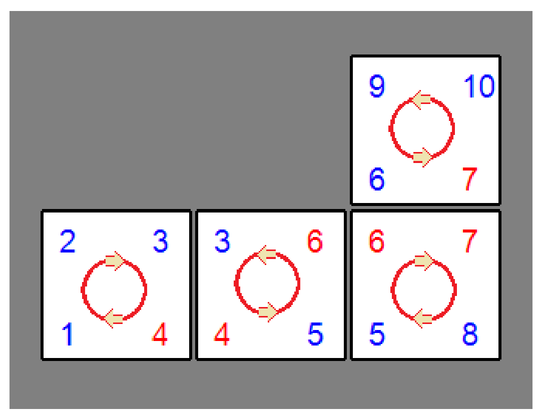
Their signatures are familiar to any of us who ever played Tetris.





This is the “aha” moment that led me to come up with connection between tetraflexagons and polyominos almost sixty years ago after reading Martin Gardner’s original edition of ‘Second Scientific American Book Mathematical Puzzles and Diversions’. Being able to visually enumerate the varieties and seeing this apply to hexaflexagons as well was the breakthrough to my method for explaining the number of varieties of flexagons with a given number of faces. I would like to know if this line of thought is mathematically valid, but it does seem to work and is easier to visualize.

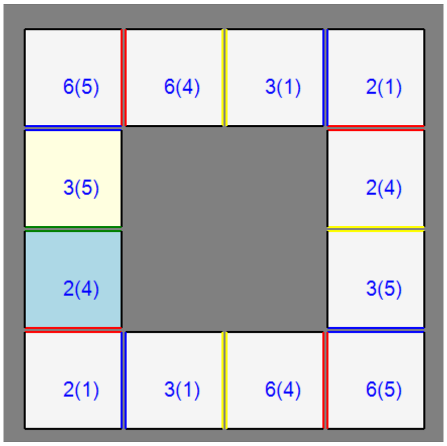




The square signature above is where the first difference shows up as it is actually a U shape and thus there is a way to add a square on top of another in the signature by adding a pair of faces on the 9-4 edge, which does work, thus making for sixteen varieties of dodecatetraflexagons instead of 12 for the number of pentominos (the U, being asymmetric, has four distinct ways to add another square, three of which lead to the “P” pentomino but have distinct flexagons to match. The maps for the five of these, and a few higher order flexagons are in the tetraflexagon catalog.

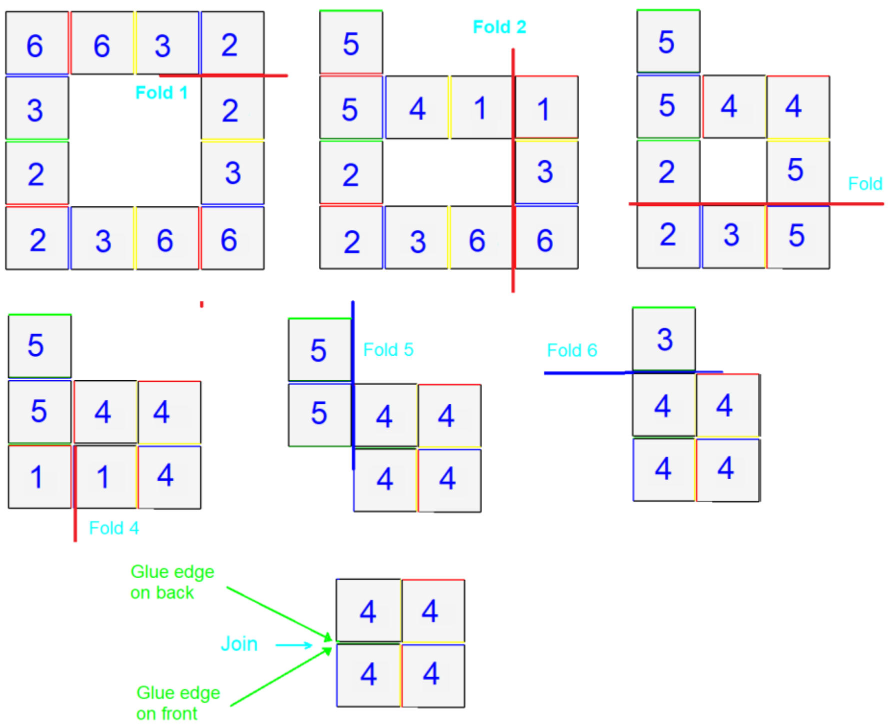
I claim that this process can produce all possible true tetraflexagons.
Maybe? I think so, still.

Right and Left handed forms of flexagons, maps, and signatures are possible. I claim that if the signature is the same, the map will be the same, regardless of the order in which faces get added.
Because the primordial tetraflexagon does not have three squares in a row in its map, and the process for generation tetraflexagons with more faces never produces three squares in a row in its maps.



So what to do with the square of squares (and its relatives)? It works, but it is sort of wonky and has a flex cycle where it half opens and has a configuration with multiple squares on both sides and won't open. If joint is pulled on, the whole thing comes apart and turns back into the square of squares. Flat. But it does work, and its flex cycle is the same as my hexatetraflexagon that is not flat, but has two or more twists in its chain of squares.

I adapted the method for folding the seamless square of squares to Martin Gardner's square of squares and it produced a working hexatetraflexagon identical in its signature from the one I discovered in 1961.



This is what it should look like after each fold as the flexagon is assembled. Note that folds 2 and 3 are both joints at once.

The joints in the map follow the same principle, red is one-way up, blue is one way down, and yellow is two way.
In this map, adapted from Martin Gardner's map, the split is on the two-way joint on the left. In these folding instructions, I am showing only the face up number you should see if the faces are numbered. I recommend numbering the faces with Post-it® notes.

Making Flexagons

The process for making flexagons is fairly straightforward. Fold the faces together in reverse order, two at a time, and make sure that the partially folded map looks like one of the maps with fewer faces.

The final primordial tetraflexagon is folded by folding the 1s and 3s together so 2s are on the front and 4s on the back. Fold the 2s together, where they meet needs to be joined with a one-way joint facing in. The easiest way to do this is carefully open the flexagon to the 3 face, turn a quarter turn to the 2 face which will now have its two halves neatly divided and open to be joined. This last joint can never be automatically perfect, but if you can align the last edge for the one way joint, the whole flexagon will be the best it can be.

All of my maps will end up with this step. (The square of squares does not)

A good flexagon should be able to be flexed through all of its faces repeatedly without wearing out, coming undone because of pulling the difficult folded stacks past each other in certain cycles, usually the 1-2-3-4 cycle because the two-way joints next to the "4(1)" faces, present in all my maps, has to fold all the other folded faces at some point to complete the 1-2-3-4 cycle.

Mathematical flexagons have zero thickness material that is inflexible and joints that are perfectly flexible, thus they fold together neatly with no need for engineering, real flexagons with more than 8 faces need more serious engineering.

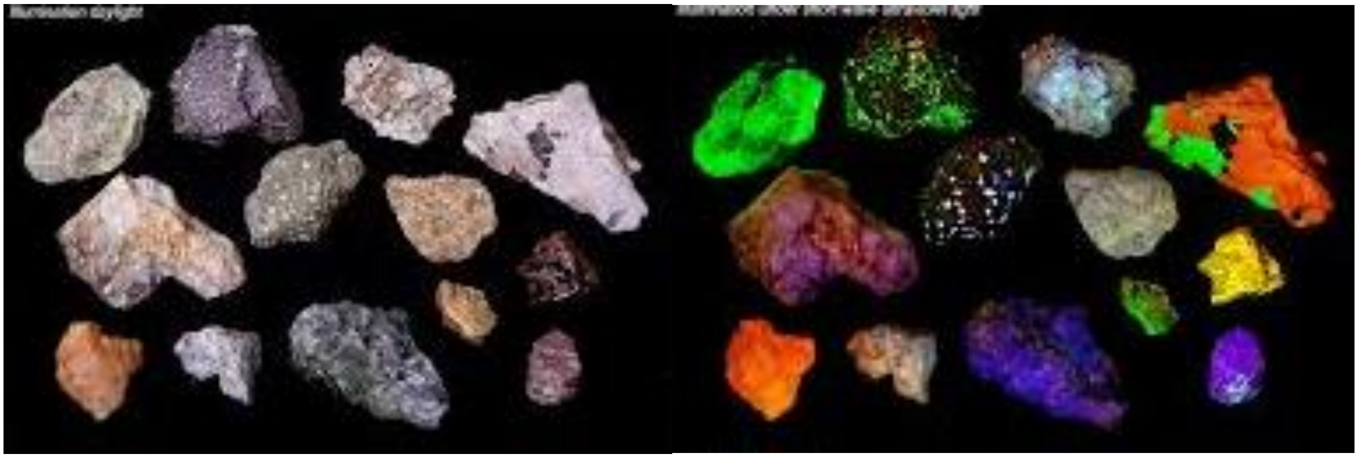
I have made nice "show" flexagons with four inch squares (and triangles), but I chose 2 inch squares and maps that could be printed on a single sheet of paper as the process for making skinned flexagons with cardboard insides and flexible joints is slow, expensive and error prone still. Even with exact printing, cutting and folding errors can ruin a paper flexagon, and any flexagon with more than 6 faces is going to have "difficult" cycles where it has to be carefully coaxed to open to make sure nothing tears or breaks.

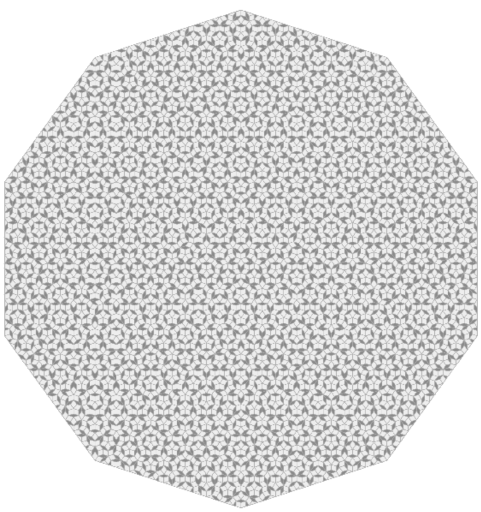
Penrose Patterns

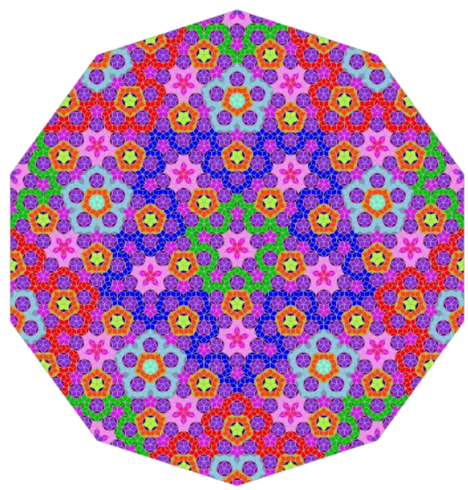
I have chosen to use my Penrose Patterns for decorations

I have written my own app that generates these patterns so I could implement my coloring system, which takes note of the fact that Penrose patterns form rings of shapes (kites, darts, skinny and fat rhombuses). My app counts the number of shapes in each ring and then assigns a color to each distinct count and colors rings of that could with the color.

Like when a black light shining on certain rocks, reveals patterns, so when the color method is applied to a random patch of the pattern, its structure is revealed and is the basis for all of my Penrose art.







The two color patterns come out of the box for free as skinny and fat is a property of every triangle.

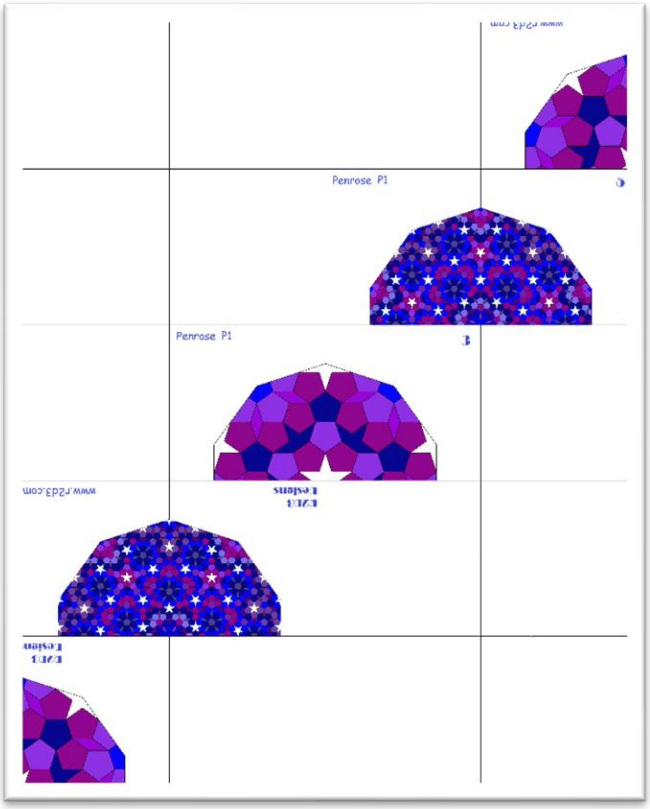
My app assigns a distinct ID to each triangle, so it can find adjacent triangles (halves of the familiar rhombuses, darts or kites) and has a database of distinct vertices, including the center points on the triangles so it can count the rings.

The algorithm for dividing triangles to make the Penrose patterns is described in Wikipedia under “Penrose Tiles”

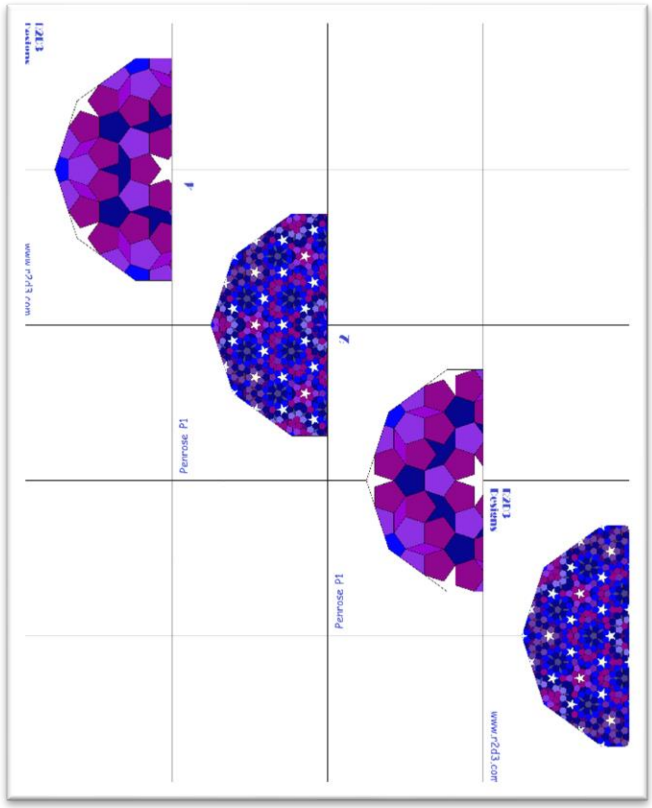
Having my own program has allowed me to bend the rules, make spheres, as well as all kinds of artworks.

Decorating Flexagons with Images

If you want the flexagon to have all four faces form an image like Markus GÖtz’s folding puzzle, the orientation matters. These are the front and back of one of the flexagons in the gift exchange as they are printed. The flexagon has to be cut and the print has to be two sided and the sides must match up.



Side 1



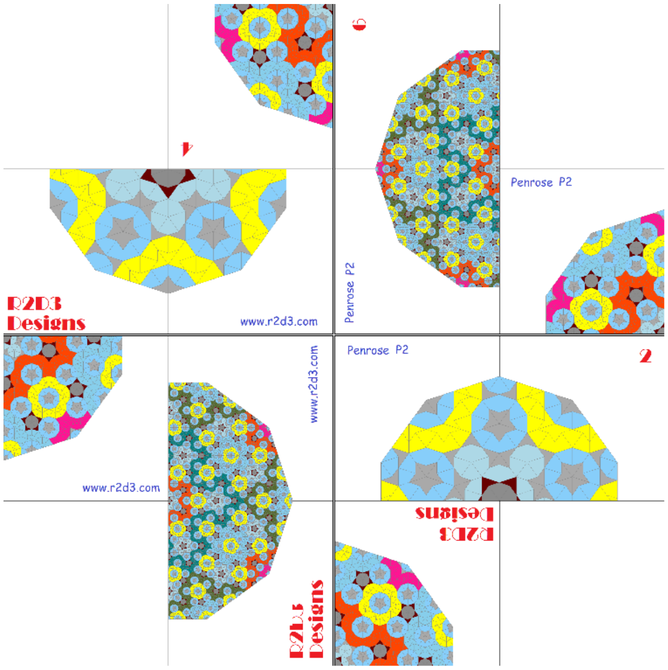
Side 2

This one shows the P1 Penrose pattern, which is the first one discovered by Sir Roger Penrose.
This is for the tetratetraflexagon, the primordial one of its kind

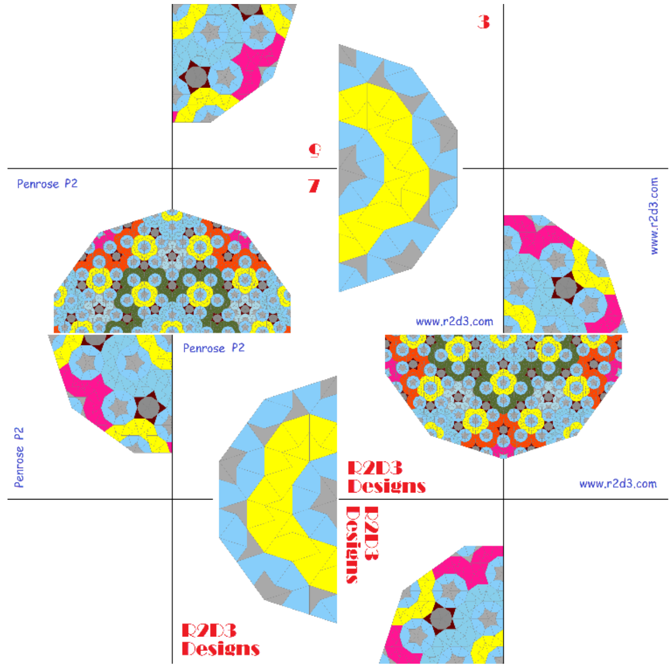
This pattern is designed to be rotated counterclockwise except for the last flex in the cycle when you rotate clockwise (so the images stay right side up.)   If the image is not a decagon, you are looking at the “back side”.   Turn it over and turn it right side up and it will show its faces in order.

The hexatetraflexagon is the first kind I discovered in 1961

This decoration was chosen to represent the P2 pattern, the one that Martin Gardner used in is chapter on Penrose Tiles.



Side 1



Side 2

In order to make a nice flexagon with more than four faces, the “outside” edges that are on the “inside” of the pattern need to be trimmed a bit, and a bit of extra spacing for the two way joints.

I recommend creasing the two way joints of a paper flexagon well before final folding.   As you fold the faces in pairs in reverse order, make sure the image aligns properly and your flexagon will behave reasonably.   Newly made flexagons need to be “broken in” by flexing them through all their cycles and making sure that the stacks set so they don’t bind and press the folded flexagon and the irregularities will work themselves out (within reason)

Errors are additive and even if the flexagon is perfectly printed and cut, small folding errors can lead to edges poking out or the images not merging.   If the two sides are not exactly matched, the odd or even faces will always be a bit off as all the even faces are on one side and the odd faces on the other.