300+ Digits of π From an (Almost) Ordinary Deck of Cards

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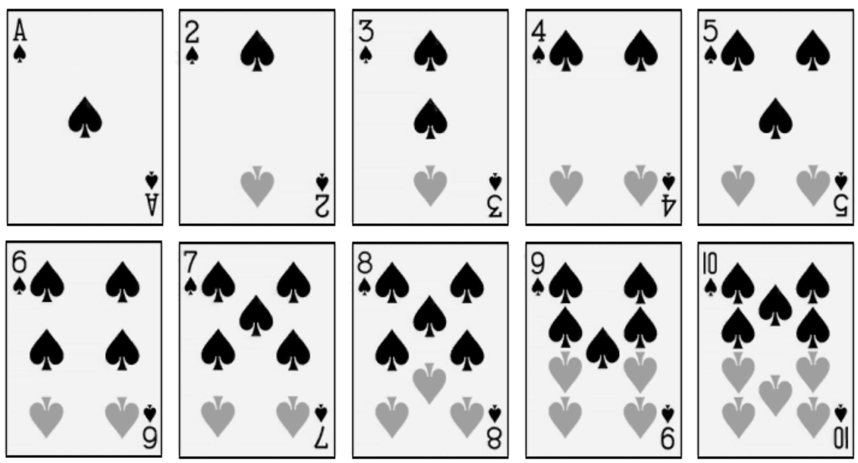
In this paper we discuss a new number puzzle involving a standard deck of cards, one that turns out to be sufficiently difficult that it is initially not clear whether the desired construction is even achievable.

Preliminaries. In this puzzle we're going to use all 52 cards in a standard deck, which is comprised of 40 number cards (A through 10 of each suit) and 12 face cards (J, Q, K of each suit). The number cards have some special features that are important to our puzzle, so the A through 10 of spades are illustrated below.



The number that appears in two corners of each card (paired with a small suit symbol) is known as the *corner index*, and we'll call this number (1 (= A) through 10) its *value*, denoted by v. The suit symbols in the middle of each card are the *pips*. Note that some pips are rightside up and some are upside down; in the graphic above the upside-down pips are illustratively colored gray. The orientation of the pips shown here is the de facto standard for a deck of cards.

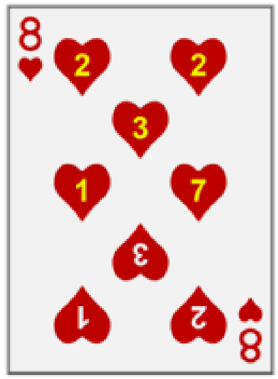
We can summarize these pip orientations by listing the *split* for each possible value, a pair of numbers (r, u) that specifies how many rightside-up pips (r) and upside-down pips (u) there are, where $r \geq v$ and $r + u = v$. The split numbers for v = 1 to 10 are shown in the table below. The cards shown above are oriented with the larger set of pips, corresponding to r, at the top.

Value (v)	1	2	3	4	5	6	7	8	9	10
Split (r,u)	1,0	1,1	2,1	2,2	3,2	4,2	5,2	5,3	5,4	5,5

Note that the 2, 4, and 10 cards, and only those, have $r = v$, so these cards look exactly the same when rotated by 180 degrees, but all the other cards are rotationally non-invariant. There are two distinct ways to place an A, 3, 5, 6, 7, 8, or 9 on a table: with the r pips facing up, or rotated by 180 degrees with the u pips facing up.

The numbered cards in a single suit have 1 + 2 + ... + 10 = 55 total pips, so the number of pips in all four suits is 55 x 4 = 220.

We now introduce the idea of *labeling* the pips. Imagine a single decimal digit (any digit 0 to 9) written inside each pip, with the orientation of each digit matching the orientation of the pip, so that if the pip is rightside up then so is the digit. Pips of the "up" and "down" orientation will be labeled with two different colors. We use the colors white and yellow, since these are both nicely visible when written inside either black (spades and clubs) or red (hearts and diamonds) pips. Here is an example of a pip-labeled card:



For the purposes of our puzzle, we're going to "read" the rightside-up part of each pip-labeled card as a sequence of decimal digits, by reading the corner index number first followed by the digits inscribed on the rightside-up pips in raster-scan order. The "10" index number on a ten card is read as the decimal digit "0", and an "A" is read as the digit 1. So the card above is read as "8 2 2 3 1 7", the 8 coming from the index number in the corner and the 22317 from the five yellow-numbered pips. Rotated 180 degrees this card becomes 8 2 1 3, from the 8 in the corner and the 213 on the white-numbered pips.

Note that, when numbering an asymmetric card (not a 2, 4, or 10), you can choose which "half" (*r* side or *u* side) gets the yellow numbers. The example above has the yellow numbers on the *r* side, but either way is acceptable when choosing how to number the pips.

If all 40 number cards are placed on a table in some order, face up, with all the pip labels of the same color on top, we refer to this as a *deal*. By definition the yellow numbers are on top in the first deal and the white numbers are on top in the second. Reading off the rightside-up digits (index number + pip numbers) of all the cards in the *k*th deal produces a sequence of n_k digits, for $k = 1$ and 2. Note that n_1 and n_2 need not be equal, but there are 220 pips and 80 corner indices, each of which contributes a digit, so the number of digits in both deals ($n_1 + n_2$) is 300.

Face cards. There is no straightforward way to interpret the indices of the face cards as decimal digits – especially since there are only three distinct indices – so we will ignore the indices on face cards but do a different kind of pip numbering, by placing zero to four digits in each of the two large pips which traditionally appear at upper left and lower right of the face card picture area. We picked four as a somewhat arbitrary upper limit by judging that it seems reasonable to put up to four digits in each large pip, but any more than four starts to look too crowded.

Here is an example of a pip-numbered face card:

In this orientation the card is read as "2 8 4" (we read these digits in scan-line order, so it's "2" from the top line then "8 4" from the second line). Rotated, it reads as 9 2 5 9. Again there is the concept of a first and second deal for the face cards, with yellow digits facing up in the first deal and white in the second, and we denote by f_k the number of digits contributed by all the face cards in the *k*th deal. Since we allow 0 to 4 digits in each large pip, each f_k is in the range 0 to 48, with the 48 achieved when a deal has 4 digits in the upper left corner of all 12 face cards.

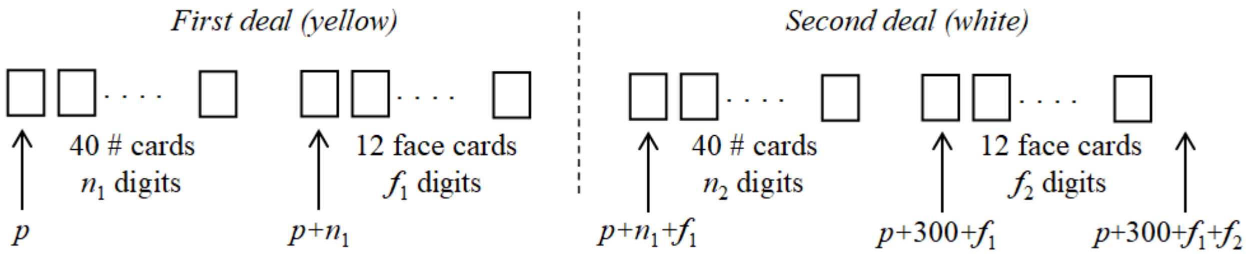
A *full deal* consists of laying out all 40 number cards followed by the 12 face cards, with all digits of one color facing up. The *k*th full deal (k = 1 or 2) produces a total of $n_k + f_k$ digits, and both deals together generate a total of $T = n_1 + f_1 + n_2 + f_2 = 300 + f_1 + f_2$ digits. Because $0 \leq f_k \leq 48$, T ranges between 300 and 396, depending on how many digits are inscribed on the face cards in each of the two deals.

Puzzle statement. Take a deck of cards and label each of the 220 pips of the 40 number cards with a single digit of your choice, with the orientation of the digits matching the orientation of the pips, and with the two digit orientations on each card colored yellow and white as described above. On the asymmetric cards you can choose which "half" (*r* side or *u* side) gets the yellow numbers, then use white for the other half. Also inscribe 0 to 4 digits in each of the two large pips of each of the 12 face cards.

Make the first full deal of the 40 number cards (in any order) followed by the 12 face cards (also in any order), with the yellow numbers facing up on every card. Read off the index number and the yellow digits of each number card in order, and the yellow digits on the face cards, and write them all in sequence. Gather up the cards and rotate the whole deck by 180 degrees so that the white numbers are on top, and again order the number and face cards any way you wish. Deal the second full deal of number cards followed by face cards. Read off all the digits again and concatenate them to the first long digit sequence. The result is a sequence of 300 to 396 digits generated by two deals from the same deck of cards. The puzzle is:

> **Can we find a two-color labeling of a 52-card deck as described above, and an ordering for the first and second deal, so that the two deals generate a pre-specified sequence of digits, such as, say, <u>the first 300+ digits of the number π</u>?**

Before presenting some solutions to this puzzle a few definitions and remarks are useful. To begin, here's a recap of the two-deal structure:



Recall that $n_1 + n_2 = 300$ and $0 \le f_k \le 48$, which means $0 \le f_1 + f_2 \le 96$. The total number of digits present in both deals is $300 + f_1 + f_2$, a number in the range 300 to 396. The values along the bottom (p, $p + n_1$, etc.) specify where we are in the digit sequence we're trying to "spell" at different points in these deals, where the whole thing starts at the pth digit of the sequence.

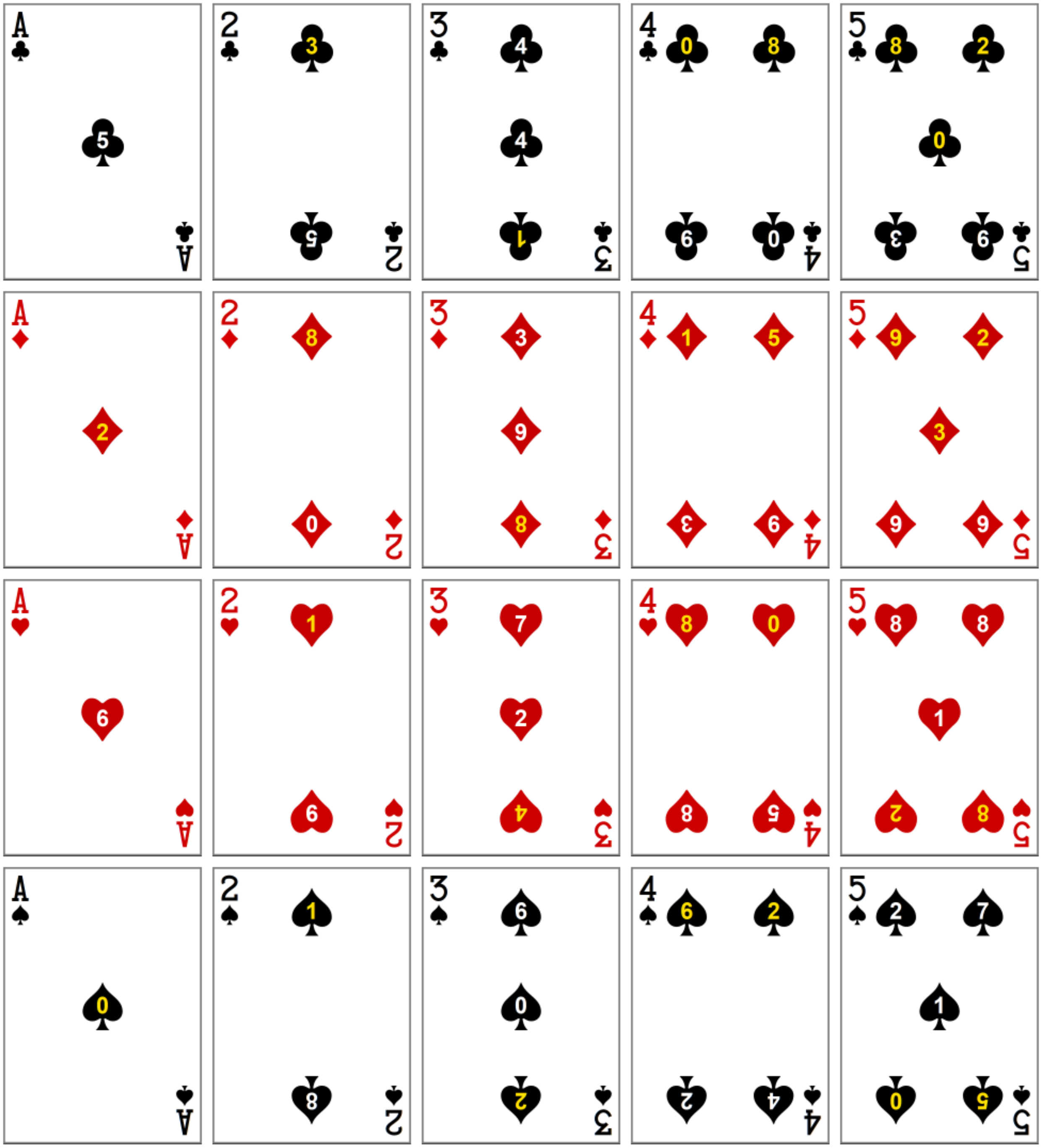
The hard part of this puzzle is ordering and numbering the pips of the number cards (the first and third sections of the diagram above) since we have to interleave the fixed index numbers on these cards with the numbers inscribed on the pips. However, since the indices on the face cards aren't used, the digits inscribed on them are essentially "free" digits. Indeed, without loss of generality we can always order the 12 face cards in both deals as, say, J,Q,K of clubs followed by JQK of diamonds, hearts, and spades.

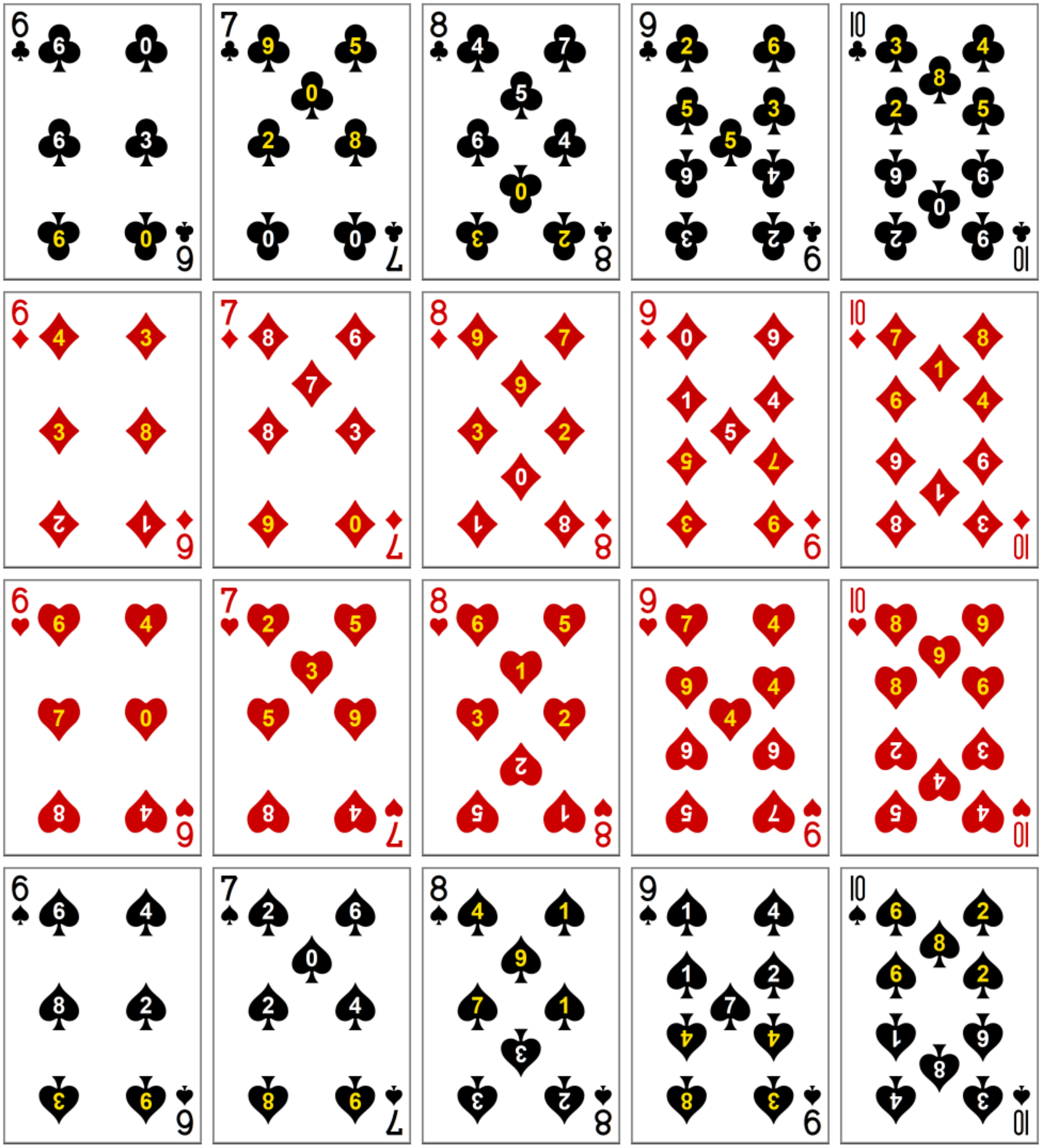
But one aspect of numbering the face cards is quite important: the value of f_1. The second deal starts at digit $p + n_1 + f_1$, which depends on f_1, so the alignment (with respect to the digit sequence) of the second deal of number cards changes when f_1 changes. This can be crucial in determining whether the number-card part of the second deal can be successfully constructed.

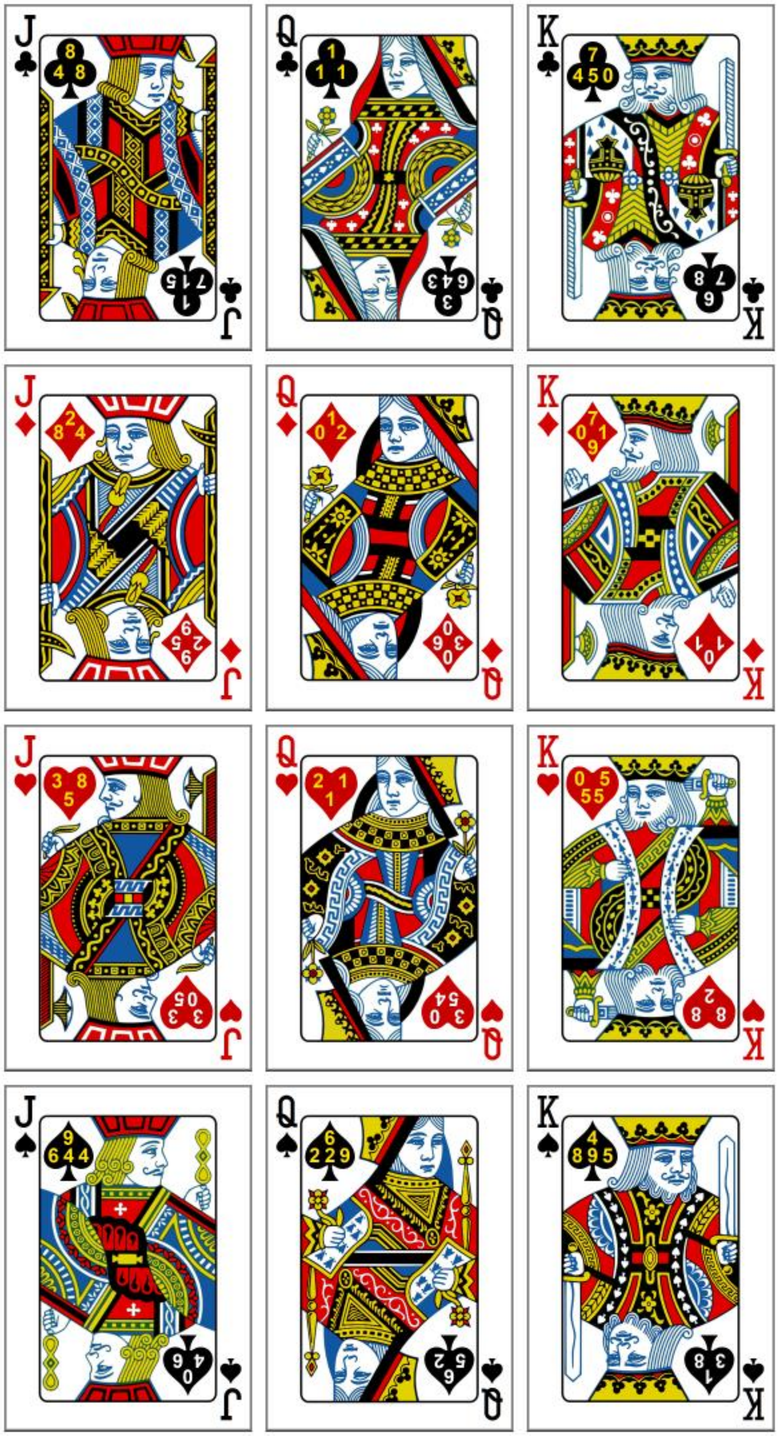
In general f_1 and f_2 can be any number from 0 to 48, for a total digit count of 300 to 396, but we realized that a digit count of 384 would be especially nice, since two different 384-card decks (if they can be constructed) could be used to span the first 2 x 384 = 768 digits of π. As all true π fans know, the remarkable run of digits "999999" ends at the 768th digit of π, so this two-deck set would be quite an elegant construction, with its digits terminating at that famous spot.

A 384-digit deck has $f_1 + f_2 = 84$, so there must be an average of 84/12 = 7 digits per face card. This can be pleasantly achieved by putting, on each face card, 4 digits in one of the two large pips and 3 digits in the other large pip. We decided to restrict our search to this special case, with $36 \le f_k \le 48$, $f_1 + f_2 = 84$, $n_1 + n_2 = 300$, and a total deck size of exactly 384. A nice subcase occurs when $f_1 = f_2 = 42$; we refer to this face-card allocation as being *balanced*. Note that the number cards also may or may not be balanced: $n_1 + n_2$ is always 300, but n_1 and n_2 may not be equal, and typically aren't. If $n_1 = n_2 = 150$ we say the number cards are balanced. If both the face cards *and* number cards are balanced we call such a solution *perfectly balanced*.

Here, now, is a successful construction of a 384-digit deck for the digits of π. The 52 cards of this deck are shown on the following three pages. Number cards in this display are oriented with the *r* side up, which means that each card can have either the yellow or white digits on top, depending on how they were assigned.

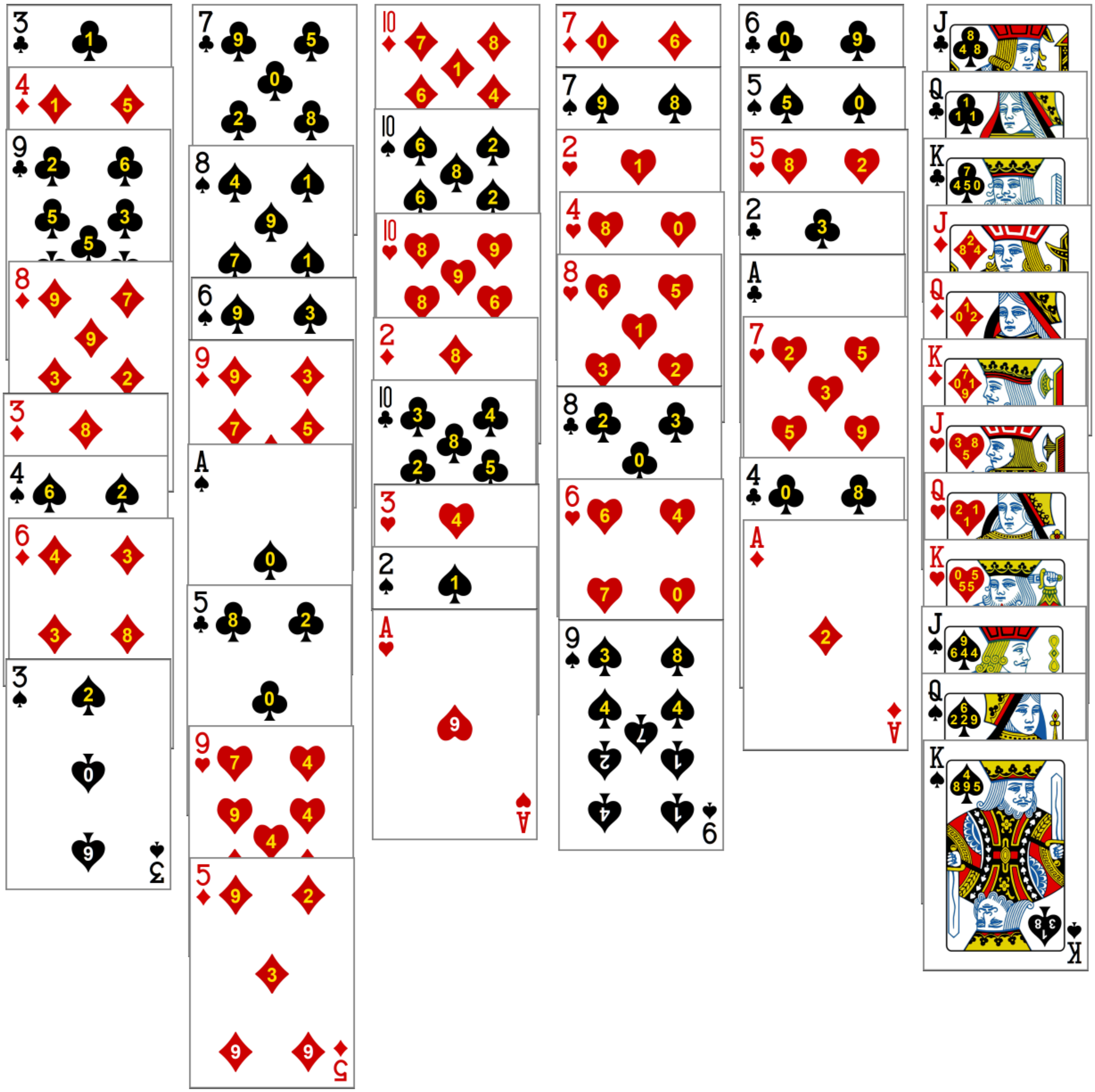






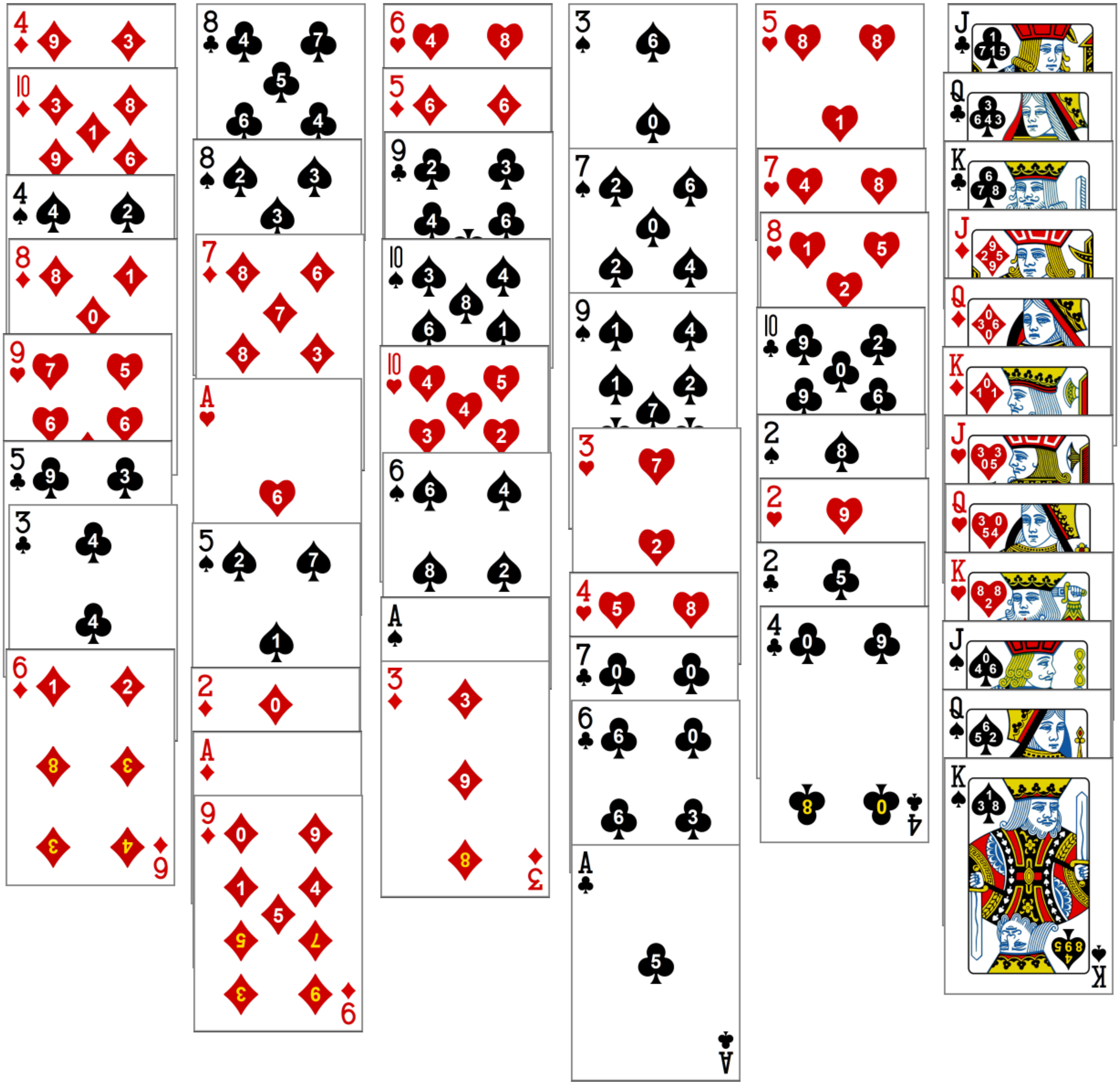
The next two pages show both deals: the first deal with all yellow numbers facing up, the second with white numbers. By dealing the cards in columns we can hide all the upside-down pips (except on the card at the bottom of each column) and directly read off the digits of π by just scanning down each column starting with the left one. The actual digits of π are displayed at the bottom of each page. Amazingly, this solution is perfectly balanced, with 150 number-card digits and 42 face-card digits in both deals, and it the *unique solution* with this property.

First Deal



Number cards 150 digits	3141592653589793238462643383279502884197169399375l 05820974944592307816406286208998628034825342117067 98214808651328230664709384460955058223172535940812
Face cards 42 digits	848111745028410270193852110555964462294895

Second Deal



Number cards 150 digits	49303819644288109756659334461284756482337867831652 71201909145648566923460348610454326648213393607260 24914127372458700660631558817488152092096282925409
Face cards 42 digits	171536436789259036001133053054882046652138

Finding solutions. Suppose we try to find a solution to this puzzle by hand, and consider just the number card deals. The first card of the first deal has to be a "3", since its value must match the first digit of π. But now we have a choice of orienting this card with the two *r* pips facing up, which will get inscribed with the next two digits (1, 4), or with the one *u* pip facing up, which will get the single digit (1). In the first case the next card used must be an Ace, to capture the next "1", but in the second case the next card must be a 4, since the only digits captured so far are (3, 1). In general, both orientations have to be tried for every number card except for the rotationally symmetric 2's, 4's, and 10's, which is a total of 40 – 12 = 28 binary choices. During the second deal, the orientation of each card is further constrained by how the cards are oriented in the first deal, but quite a few still have to be tried in both orientations.

The number of branches in this search tree is too enormous for a search by hand, so we wrote a computer program that does an exhaustive, recursive, depth-first search for solutions for the first number-card deal, and then for each successful first deal does a similar search (whose starting point in the digits of π depends on the value of f_1) to see if the second number-card deal can also be constructed. The basic recursive task in this algorithm is to pick one more card from the deck, choose its orientation, add it to a tentative solution, then recurse. The value of this card must correspond to the next unused digit of π, but we usually have to try both orientations of the card, which determines whether it uses up the next r+1 or u+1 digits of π.

Recall that a 384-card deck has $36 \le f_k \le 48$, so there are 13 different choices available for f_1. We try all of these values in the order 42, 41, 43, 40, 44, etc., and stop at the first solution (if any) found by the algorithm described above. This finds a solution that's as close to balanced (f_1 = 42) as possible.

After success with the first 384 digits of π we ran the same search using the next chunk of 384 digits in π (i.e., digits 385 to 768). We found solutions for f_1 = 37, 41, 45, and 47 combined with n_1 = 147 or 149. While this is interesting, neither the number cards nor face cards are balanced in these solutions. We're greedy, and wanted a solution for the second deck that's perfectly balanced, so we wondered if there might be another degree of freedom we could use to help find a perfectly-balanced 384-digit deck for digits 385-768 of π.

Alternate Splits. There is, indeed, a subtle trick that can be employed to significantly enlarge the search space for finding solutions. Recall that the split into rightside-up and upside-down pips on each number card is determined by the traditional orientation of the pips in a standard deck, as shown in the diagram on the first page of this paper. Since we insist that rightside-up digits always go on rightside-up pips, we must always follow the (r, u) splits as shown in the table on page 1.

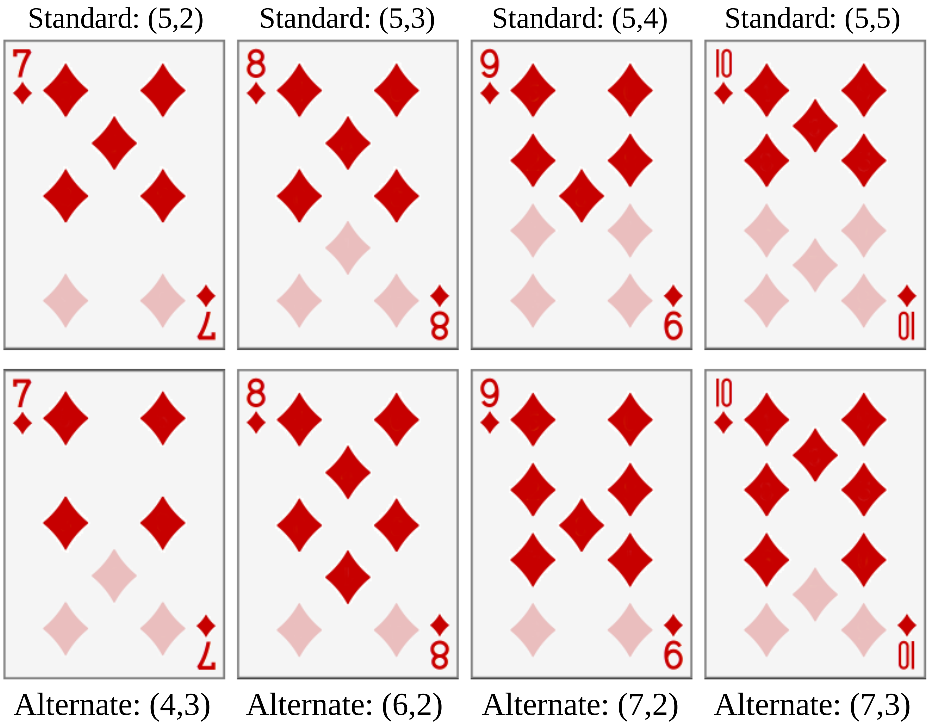
Strictly speaking, however, this rule only needs to be followed for the club, heart, and spade cards, since their suit symbols have a concept of "rightside up". This is not the case for diamonds, whose symbol is invariant under a 180-degree rotation. So we could, in theory, split the rightside-up and upside-down pip numbers differently on the diamond cards, and this will not cause any unwanted appearances of an upside-down digit on a rightside-up pip. We call these alternate ways of dividing the pips on the diamond cards *alternate splits* (AS for short), as opposed to the *standard splits* defined by the pip orientations of a traditional deck of cards.

What alternate splits are possible? For aesthetic reasons, we insist on these two rules:

(1) The pips must be split by a horizontal line that runs the full width of a card. This means, for example, that 4 cannot be split as (3, 1).

(2) Both numbers in the split must be nonzero. So, for instance, 4 cannot be split as (4, 0).

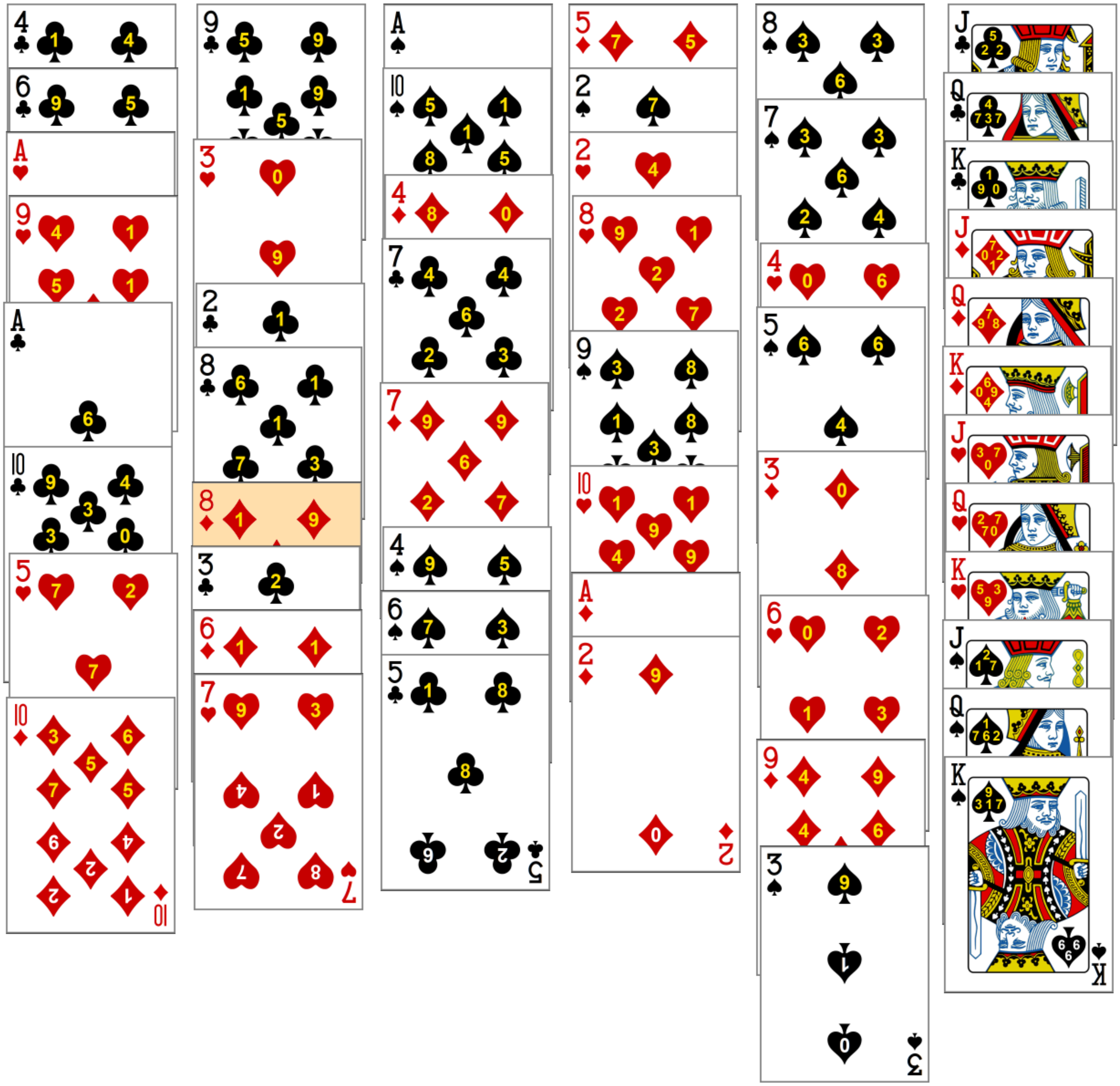
These two rules mean that there aren't any alternate splits available for the A, 2, 3, 4, 5, and 6 cards, but the 7, 8, 9, and 10 cards do have alternate splits, as shown below. The cards in the top row are shaded to show the standard split, with the alternate versions depicted on the second row.



There's actually a second split available for 10, which is (8, 2). We decided to not allow this one, as it is simpler and cleaner to have a single AS choice for each value (7, 8, 9, and 10). This means that there are exactly four cards in the whole 40-card deck (the 7, 8, 9, and 10 of diamonds) on which a unique alternate split can be used, if desired, to help achieve the successful construction of a solution. Since we can either use or not use the AS version of each of these four cards, there are 16 different AS configurations. So the full process of finding a solution now is to run the exhaustive search described above for each of these 16 AS choices.

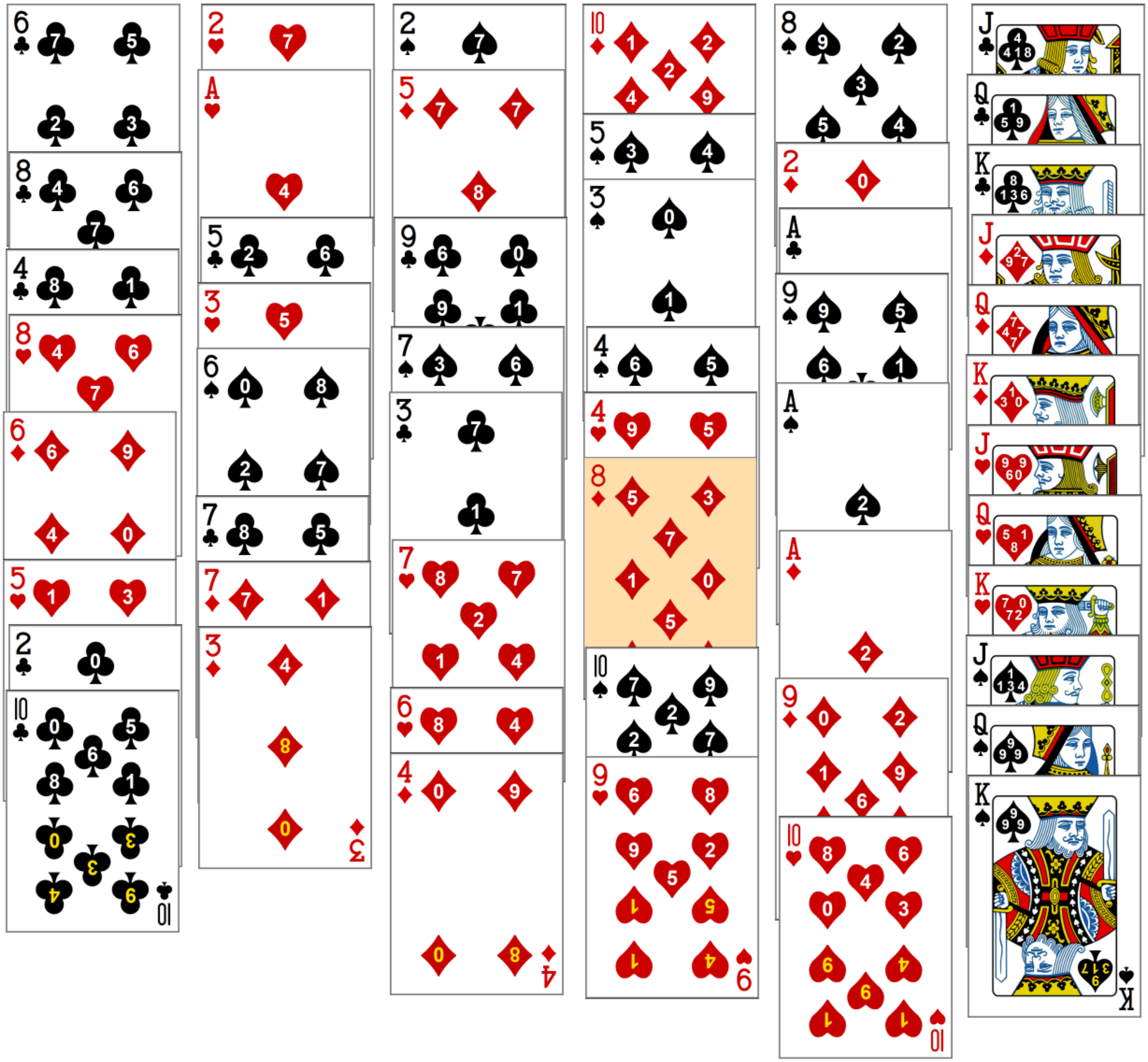
We say that a solution using zero AS cards is *pure*. As already mentioned, there is no pure, perfectly-balanced solution for digits 385-768 of π, but there *are* some perfectly balanced solutions using AS cards. The two solutions with the fewest AS cards use just one: either the 8 or 10 of diamonds. The 8-of-diamonds solution is shown on the next two pages. (To save space only the two deals are shown, since the full numbering of all 52 cards can be inferred from the two deals.) The alternate 8 of diamonds, with its (6,2) split, is colored beige to make it easy to spot in both deals.

Second deck, First Deal:



Number cards 150 digits	41469519415116094330572703657595919530921861173819 32611793105118548074462379962749567351885752724891 22793818301194912983367336244065664308602139494639
Face cards 42 digits	522473719070217986094370277053921717629317

Second deck, Second Deal:



Number cards	67523846748184676694051320005681271452635608277857
150 digits	71342757789609173637178721468440901224953430146549
	58537105079227968925892354201995611212902196086403
Face cards	
42 digits	441815981362977477130996051870721134999999

Note the 999 and 999 on the final two face cards, encapsulating the famous 999999.

Continuing through the digits. What happens if we keep marching through the digits of π in 384-digit chunks? Can we always find a solution, or do some 384-digit groups occur for which no solution exists for any value of f_1 (in range 36 to 48) and any set of AS cards? To get some idea of what happens we looked for solutions for the first 261 384-digit chunks of π spanning 261 x 384 = 100,224 digits, which took roughly 200 minutes of runtime on a single core of a 10-core 2022-era PC (or 20 minutes using all 10 cores in parallel). Within this range, there are four places (starting at digits 14208, 38400, 57216, and 88704) where no solution exists. Is there some way to handle these troublesome spots?

Recall that these 384-digit decks have these restrictions:

[a] $n_1 + n_2 = 300$
[b] $f_1 + f_2 = 84$
[c] $36 \leq f_k \leq 48$,

We do not want to give up condition [a], but conditions [b] and [c] can be relaxed to:

[b'] $0 \leq f_1 + f_2 \leq 96$.
[c'] $0 \leq f_k \leq 48$

which was our original formulation, with a deck spanning from 300 to 396 digits, prior to fixing the deck size at 384 digits. So let's distinguish between a *384 deck*, satisfying [a], [b], [c], and a *general deck* satisfying [a], [b'], [c'].

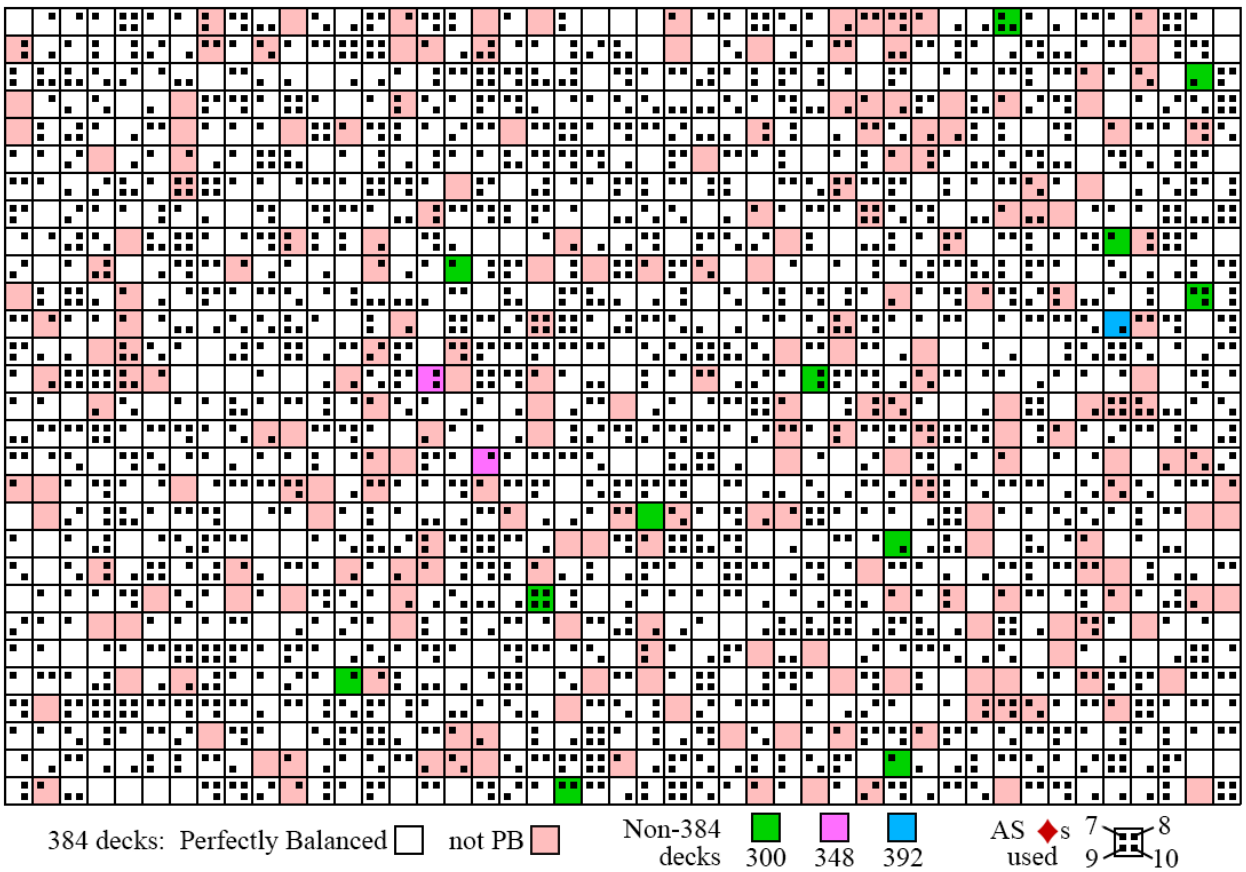
A possible strategy for getting past impossible positions in the digits of π is:

(1) Use 384 decks by default, marching through the digits 384 at a time.
(2) When a position p is reached where no 384 deck works,
  (2a) Move back to position $p - 384$.
  (2b) Try general decks with various values of f_1 and f_2 to (hopefully) find a solution. Let d (*not* equal to 384) be the number of digits in a deck that works here.
  (2c) Continue from position $p - 384 + d$ using 384 decks.

Since $d \neq 384$, $p - 384 + d \neq p$, so in step (2c) we're trying to find a 384-deck solution in a position different from the position, p, where the 384 deck failed in step (2). By trying various values of d in step (2b) we can hopefully find one that makes step (2c) work. There are 96 different values of d that can be tried: every integer from 300 to 396 except 384.

Using this strategy we were able to find a series of decks that encode the first half million digits of π (500,048, to be exact). In step (2b) we first tried a 300-digit deck (which basically dispenses with the face cards), since this provides the largest shift in position (-84) within the digit sequence, by which we're hoping to overcome the "bad" position p. If a 300-digit deck didn't work we next tried 348, then 392. Within this 500,048-digit range we never needed to try other values, meaning that we only used 3 of the 96 d values available.

The figure below shows the result of our 500,048-digit search, where each block represents one deck in a series of 45 x 29 = 1305 decks. For each deck we first attempted to find a perfectly-balanced (PB) solution, while also minimizing the number of AS cards. If no such solution exists then we looked for a non-PB solution. To distinguish them, perfectly-balanced 384 decks are colored white, while non-PB 384 decks are pale red.



Only 15 non-384 decks (of just three varieties) were required. These are colored green, magenta, or blue, as shown in the legend, according to their digit count. Although not indicated in the figure, 14 of these 15 solutions are perfectly balanced, the only exception being the single 392-digit deck. Overall, 1078 (82.6%) decks are perfectly balanced, and 125 (9.6%) of them are perfectly balanced *and* pure – including, as we have already mentioned, the very first one.

The AS cards used in a solution, if any, are indicated by one to four dots inside the square. The figure at the right in the legend shows which dot positions represent the 7, 8, 9, and 10 cards.

All of π? Does an infinite sequence of contiguous labeled decks exist with which to spell out *all* the digits of π? If the π-is-normal conjecture is true, the answer is no:

> **Theorem:** *If π is normal, any contiguous sequence of general decks (each spanning at least 300 and at most 392 digits) will eventually fail – i.e., will encounter a section of π's digits where no general deck can be constructed.*

Proof: Define a *bad window* of digits in π as a contiguous block of *n* digits of π which has the following property: The 40 number cards in a deck cannot be labeled in a way that allows either deal from this deck to capture the digits contained in the given window. One example of a bad window is *a block of digits in which any one specific decimal digit appears fewer than four times*. This works because there are four cards with each index number in the deck, and for each of these cards there must be digit in the window to assign it to. So if any digit occurs fewer than four times, a deal cannot be constructed.

Now recall that each deck has the structure $N_1 – F_1 – N_2 – F_2$, where N and F represent a block of number cards and face cards, respectively. The number of digits that can be captured by each N and F is bounded in size, so if the length, *n*, of the bad window is large enough then either N_1 or N_2 of *some* deck (in the sequence of decks) must lie entirely within the bad window, and cannot be constructed. The theorem follows by noting that if π is normal, arbitrarily large bad windows of the type described above are guaranteed to exist. ■

Bad window size. Exactly how large does the bad window need to be? The most digits that a number-card deal can capture occurs when the *r* pips of *every* number card are used in the deal. The sum of the ten *r* numbers (for A to 10) is 1 + 1 + 2 + 2 + 3 + 4 + 5 + 5 + 5 + 5 = 33, which multiplied by 4 for the four suits gives 132, plus 40 for the index numbers = 172. But we can increase this a little more by using the alternate split on the 8, 9, and 10 of diamonds (but not the 7, since the alternate split actually reduces the value of *r*); this changes the final 5 + 5 + 5 in the sum to 6 + 7 + 7, for a total of 177.

Now consider two number card deals with a face card deal between them (N – F – N). The most digits this can represent is when both N's are 177 and F = 48, so 177 + 48 + 177 = 402. If the bad window is one smaller than this (401), then no matter how the 402 digits of N – F – N line up with it, *at least one N will lie totally within it*, and therefore be impossible to construct.

The location of a specific 401-digit bad window in π is not yet known. We searched the first 100,000,000 digits and found that the longest one is this 264-digit specimen at digit 6,562,558:

55219456178142178562058161430560084829194894522917
65224987912952876682978117724669017646018271765886
51349759408824181279876983955661018207966027682609
69925986952754875228992744105286487475109745400419
66491666472167120896527642127106288745970106469107
72458186210661

Note the three 3's shown in red, the only 3's in this whole group of digits. Also note that 264 is still a long way from 401! The question of how far we can continue this deck-building game in the digits of π before provably getting stuck remains an open problem.