Magic and problems from half a millennium ago: The recreational problems of *Tratado da Pratica D'arismetyca* by Gaspar Nycolas, 1519

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The Tratado in its context

By the end of the 15th and early 16th centuries, commercial activity in Europe had become very intense, promoting the transition from a feudal economic system to another in which trade took center stage. In this context, two types of mathematical publications appeared in Portugal: those supporting the navigations and those dealing with problems related to trade.

Regarding the field of Arithmetics, there were about 40 manuals published in Europe between 1472 and 1519. (Almeida 1994, p. 25). This proliferation may have accompanied the establishment of abacus schools, especially in Italy, after the publication of Fibonacci's *Liber Abaci* in 1202, which introduced the use of Indo-Arabic numerals and related algorithms.

In Portugal, three arithmetical treatises were printed, in Portuguese, at the beginning of the 16th century. The first one was *Tratado da Pratica D'arismetyca* by Gaspar Nycolas (Nycolas 1519), which the authors have studied in order to to produce a reedition, to appear soon, and which will be the theme of this paper. The book was first published in 1519 and saw eleven more re-editions (Almeida 1994, p. 82) until the 18th century, the last one in 1716. The author of this text prepared a modern version of Nycolas' book, to appear soon in the Fundação Calouste Gulbenkian's catalogue.[1] The other two treatises were *Prática Darismética* by Rui Mendes in 1540

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(Mendes 1540), and *Tratado da Arte de Arismética* by Bento Fernandes in 1555 (Fernandes 1555).

They all follow a similar structure, starting with the organised presentation of arithmetical procedures, either abstractly or immediately applied to practical cases. It should be noted that all of them use, from the beginning, the Indo-Arabic numerals, completely abandoning the Roman numerals and the operations performed with them, such as the so-called *conta castelhana*, still present in some Spanish textbooks of this period, and which curiously would reappear in a later Portuguese textbook, *Flor Necessária da Arismética* by Afonso Guiral and Pacheco, in 1624. The positional notation, typical of the Indo-Arabic system, greatly facilitates written algorithms which, unlike abacuses, keep the intermediate steps visible at all times, available for inspection.

Focosing now on the arithmetic by Gaspar Nycolas, we see that, after the description of this numerical writing, we move on to the four operations and the algorithms for performing them, which are similar to those we use today. The only algorithm that is considerably different is that of division: the books use the galley division.

After the description of the four operations, we move on to calculation rules, such as the rule of three, which occupies several sections, and which is presented with several variants (which can be considered implementations of the compound rule of three, nowadays abandoned because it can be reduced to iterated use of the of the rule of three). Double false position, an ancient method for solving linear equations, is also used systematically. There are several sections devoted to fractions, and to the extension of the calculation rules to cases where the data are fractional rather than integer. Then we find some sections devoted to practical problems on taxes or bartering. There are sections mainly of numerical problems, a long section on geometry, and another on methods for extracting square and cubic roots. At the end of the book there are several problems on silver alloys.

Contrary to the pedagogical principles to which we are accustomed today, the solution to these problems is mostly presented without explanation — the author begins the resolution with the expression "Do it this way" and describes the method for solving the problem (in many cases it is not immediately clear why the given resolution actually solves the problem). The motivation, clearly, was to mechanise these methods so that the reader could put them into practice expediently in the daily problems of commerce.

Alongside these pragmatic considerations, we find a long collection of problems of a recreational nature. It was already a medieval tradition, that of accompanying the mathematical textbooks by lists of problems intended to develop reasoning and to entertain. Gaspar Nycolas explicitly refers to Luca Pacioli's *Summa* as a source for these problems, but some of them come from older traditions, both medieval European and classical Greek. Some of these problems can be solved using the methods presented earlier in the book, but others need an *ad hoc* reasoning, or, as the author says, can only be solved "by fantasy", that is, by thinking of a resolution specifically for the given problem.

For the sources of problems, we searched for previous occurrences in the literature, namely in the most well-known collections, namely: Metrodorus' *Greek Anthology* (Paton 1980), Alcuinus' *Propositiones ad Acuendos Juvenes* (Hadley and Singmaster 1992), Fibonacci's *Liber Abaci* (Sigler 2002), Treviso's *Arithmetic* (Swetz and Smith 1987), Pamiers' *Arithmetic* (Sesiano 2018), Chuquet's *Le Treviso* (Chuquet 1881, Chuquet and Marre 1881), *Summa* (Pacioli 1494) and *De viribus quantitatis* (Pacioli ca. 1509, Hirth 2015) by Luca Pacioli and *Conpusicion de la arte de la arismetica y Juntamente de geometria* by Juan Ortega (Ortega 1512.

Recreational mathematics, often immersed in works of another nature, has often been little noticed, even decried, by scholars. However, it is increasingly becoming unavoidable in the historical approach. Its roots are thousands of years old and it can no longer be denied that recreational motivation is present in a relevant part of the evolution of mathematics (Singmaster 2017). Of the three arithmetic books we mentioned, it is the one by Gaspar Nycolas that devotes the most time to topics of recreational mathematics, about a third of the book.

A few selected problems

Sum and product equal

> Give me a number that is the same, either added and multiplied. You may know that there is no other whole number except 2, because 2 and 2 is 4 and 2 times 2 is 4. However, let's exclude this one, and try and find two numbers that give the same result both added and multiplied.
>
> Here's a general rule for such questions. Take any number, whichever you want, and divide it by one of its parts, whichever you want, and whatever is missing from that part, you will divide it again by that remainder, and whatever comes, these are the numbers demanded.

> For example, take 7, divide by 4 and you get 1 and $\frac{3}{4}$, this is one of the numbers. To know the other one, take the difference between 7 and 4, which is 3. Divide 7 by 3 and you get 2 and a third, this will be the other number. The sum of these numbers is 1 and $\frac{3}{4}$ plus 2 and a third, which is 4 and $\frac{1}{12}$. If you now multiply one and $\frac{3}{4}$ by 2 and $\frac{1}{3}$ you get the same 4 and $\frac{1}{12}$.

This problem starts with the following question: find a number x such that $2x = x^2$. There are only two numbers with this property, 0 and 2. The text suggests that this question would be something of a riddle, as it adds that when asked, one should immediately exclude the number 2. Next, the problem is slightly modified: two numbers are now asked such that their sum is equal to the product. The author then gives a rule for finding such numbers: write a number a as the sum of two parts $a = b + c$ and take the numbers a/b and a/c. And indeed, their sum and product is a^2/bc. The author gives no indication about the origin of this method.

Generating squares

> Give me a number that, if you take away 11 from it, it becomes a square, and if you put 10 on it, it also becomes a square.
>
> This is the method: join these quantities that you want to take and put, and from that sum always take one. Divide in the middle what remains, and that half always multiply in itself. To this multiplication you will add that amount that you want to take it out, and after all this, you'll get be the number you were asked for.
>
> In this example: add 10 and 11, you get 21. Take 1 and 20 remain, take half of that, which is 10. These I say multiply in themselves, and you will make 100. To these you must add the amount you want to take out, and you get 111. And this is the number that if you take 11 from it, you get a hundred, which is a square, whose root is 10, and if you add 10 you will also get a square number, which is 121 whose root is 11.

We are looking for a number x such that

$$\begin{cases} x - 11 & = & \square \\ x + 10 & = & \square \end{cases}$$

where $\square$ represents a generic perfect square. The procedure given by the author leads to the solution $x = 111$ ($111 - 11 = 10^2$, $111 + 10 = 11^2$).

In general, given two integers a and b, with odd sum s, one is asked to calculate

$$\left(\frac{s-1}{2}\right)^2 + a$$

Naturally, subtracting a, the result is a perfect square. If we add b to it, we get

$$\left(\frac{s-1}{2}\right)^2 + s = \frac{s^2 + 2s + 1}{4} = \left(\frac{s+1}{2}\right)^2$$

The perfect squares obtained by this method are always consecutive. Other solutions can be obtained, noting that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, and therefore the difference of two squares, possibly non-consecutive, is always the sum of consecutive odd numbers. In this case, we can write $21 = 5+7+9$, so $21 = 1 + \cdots + 9 - (1 + 3) = 25 - 4$, and therefore $4 + 11 = 15$ would be another solution. As 21 cannot otherwise be expressed as the sum of consecutive odd numbers, these are the only solutions on the integers.

A broken weight

> A man had a stone that weighed 40 *arráteis*, it hit the ground and was broken in four pieces. With these 4 pieces he produced any number of *arráteis* as he was asked for, from one to 40. Now I demand how much each of them weighed.
>
> Know that this one has no rule [...] it is made of fantasy.[2] Know that one of the pieces has one *arrátel* another has three and the other has 9, which is the square of three, and the other has 27, whose cubic root is three. But this rule is not general, it is by fantasy. So you will say that from the four numbers 1, 3, 9, and 27 the four pieces were made.

This problem appears already in Fibonacci (Sigler 2002, p. 420), Chuquet (Chuquet and Marre 1881, p. 451, CXLII) and in Pacioli (Pacioli 1494, F97, 34). However, its origin is more remote, Topfke references a Persian occurrence in the XI century (see Tropfke et al. 1980, p. 633).

Presumably the problem concerns the use of a balance scale, with two plates, in each of which you can place either merchandise or weights. Thus,

[2]Here, "fantasy" must be understood as reasoning without application of any standardised procedure.

we look for four positive integers, x_1, x_2, x_3, x_4, whose sum is 40 and such that any natural n, $1 \leqslant n \leqslant 40$, can be expressed as a linear combination

$$n = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$$

where the coefficients α_i can take the values $-1, 0, 1$. The author presents the solution, justifying it with "fantasy". Chuquet states that the sequence of weights starts with 1 and then each weight is one unit plus twice the sum of those that precede it. Pacioli starts with 1 and successively multiplies by 3, which is equivalent. Fibonacci mentions both procedures.

There is an interesting detail in the way the author poses the problem: in no other source the four weights are the result of a larger object being broken. This particular setup tells us that probably Nycolas had access to another source, possibly not so well-known as the ones we refer to.

Bags

> There are two bags, one holds 8 *alqueires* of wheat and the other holds two. Now, I unsew them and make a bag that's as tall as they were before. I ask how many *alqueires* the big sack holds.
>
> Do it this way. Combine 2 with 8 and that's 10, save these. Now multiply the bags against each other, meaning 2 times 8, which is 16, take the root which is 4, double it, and it is 8. These 8 together with the 10 that I ordered you to save and are 18, and these many *alqueires* will the big sack hold.

Let's assume that sacks are obtained from two overlapping rectangles of fabric sewn along three sides, like the so-called burlap sacks. It is natural to assume that the volume of each bag is proportional to the square of the base seam. As the height is constant in this problem, we can suppose it to be unitary and, being the measures of the base seams c_1 and c_2, the volumes will be $8 = c_1^2 k$, $2 = c_2^2 k$, for some constant k. Joining the pieces of cloth, we obtain a sack with a seam at the base measuring $c_1 + c_2$. The volume will then be

$$(c_1 + c_2)^2 k = c_1^2 k + 2c_1 c_2 k + c_2^2 k = c_1^2 k + 2\sqrt{c_1^2 k c_2^2 k} + c_2^2 k = 8 + 8 + 2 = 18$$

as in the text. Alternatively, if we consider the sacks as the side surfaces of cylinders, assuming unit height, the problem is reduced to determining the area of the circle whose perimeter is the sum of the perimeters of two circles of areas 8 and 2. The result is, again, 18.

Break 9

> Divide 9 into two parts such that, dividing the greater by the smaller, I get 19. This is the rule: add one to the number that you want to obtain, this will be your divisor, and the dividend is 9. Therefore, divide 9 by 20, and you get 9/20, this is one of the numbers. The other will be $8\frac{11}{20}$ as you can prove.

The system to be solved is $x + y = S$ and $x/y = Q$ (with $S = 9$ and $Q = 19$). From the second equation, we get $x = Qy$, and substituting into the first equation, we get $(Q + 1)y = S$, which is the solution shown.

Conclusion

The *Tratado de Prática Darismética* can be seen as a source for understanding the commercial mathematics of the 16th century, crucial for Portugal and relevant for the rest of Europe. In the teaching tradition of practical mathematics, Nycolas' text shines as a sophisticated pedagogical book. Including both pragmatic exercises, solved by application of general rules, together with recreational problems, which entertain with their appeal to imagination and "fantasy", the author offers the students in abacus schools (and, more generally, those training for commercial matters) a glimpse of abstract rigorous thought in a ludic context.

We believe that the *Tratado*, being the first book on mathematics printed in Portugal, contributed decisively to the dissemination of mathematical teaching, now aided by the printing press, which permitted a much faster and secure expansion of the subject. Its popularity can be gauged by the number of editions of the book (up until the 18th century), meaning that it became a source of mathematical learning for several generations of Portuguese merchants, accountants, and likely also people interested in a good problem to wrap their heads around.

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