# ON SOME EQUATIONS INVOLVING EULER TOTIENT FUNCTION

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## MOHAMED LOTFI

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MOHAMED LOTFI

ABSTRACT. We prove that if $n$ is a solution to the equation $s(n) = \phi(n)$, where $s(n)$ is the sum of the proper divisors of $n$ and $\phi(n)$ is Euler Totient function, then $n$ is either 2 or an odd square. For a solution $n$ that is an odd square, we show that the number of its distinct prime divisors is greater than 2 and $\prod_{p|n} \dfrac{p}{p-1} > \varphi$, $p$ is prime and $\varphi$ is the golden ratio. Finally, we prove that the only solutions to the equation $d(n) = \phi(n)$, where $d(n)$ is the number of divisors of $n$, are 1, 3, 8, 10, 18, 24, 30.

## 1. INTRODUCTION

Different functions in Number Theory have been extensively studied during the last two centuries. Among the most famous of these functions are Euler Totient function $\phi(n)$, the number of divisors function $d(n)$ and the sum of proper divisors function $s(n)$. In this paper, we will investigate the positive integer solutions to the two equations

$$\phi(n) = d(n) \tag{1}$$

and

$$\phi(n) = s(n) \tag{2}$$

1.1. **Definitions.** Given $n = p_1^{\alpha_1} \ p_2^{\alpha_2} \ ... \ p_k^{\alpha_k}$, where $p_i$ are distinct primes and $\alpha_i$ are positive integers for all $1 \leq i \leq k$, $k$ is a nonnegative integer.

We provide some definitions that are necessary for the rest of the paper:

**Function I:** Euler Totient Function $\phi(n)$.

$\phi : N \to N$ equals the number of positive integers smaller than or equal to $n$ that are relatively prime to $n$

$$\phi(n) = n \, (1 - \frac{1}{p_1}) \, (1 - \frac{1}{p_2}) \, ... \, (1 - \frac{1}{p_k})$$

which is equivalent to

$$\phi(n) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \dots p_k^{\alpha_k - 1}(p_1 - 1)(p_2 - 1) \dots (p_k - 1)$$

We will use the second definition for the rest of the paper.

Examples: $\phi(4) = 2$ , $\phi(5) = 4$ and $\phi(6) = 2$

**Function II:** Number of divisors Function $d(n)$.

$d : N \to N$ equals the number of positive integers greater than or equal to 1 that divides $n$ (including n itself), and

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$$

Examples: $d(4) = 3$ , $d(5) = 2$ and $d(6) = 4$

**Function III:** Sum of divisors Function $\sigma(n)$.

$\sigma : N \to N$ equals the sum of positive divisors of $n$ (including $n$).

$$\sigma(n) = (1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \dots + p_2^{\alpha_2}) \dots (1 + p_k + p_k^2 + \dots + p_k^{\alpha_k})$$

Examples: $\sigma(4) = 7$ , $\sigma(5) = 6$ and $\sigma(6) = 12$

**Function IV:** Sum of proper divisors Function $s(n)$.

$s : N \to N$ equals the sum of proper divisors of $n$ (i.e. the divisors of $n$ excluding $n$ itself).

$$s(n) = \sigma(n) - n$$

Therefore, $s(n) =$

$(1 + p_1 + p_1^2 + \dots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \dots + p_2^{\alpha_2})...(1 + p_k + p_k^2 + \dots + p_k^{\alpha_k}) - p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$

Examples: $s(4) = 3$ , $s(5) = 1$ and $s(6) = 6$

All the previous functions are multiplicative, except for $s(n)$. For more information and additional results we refer the reader to [1] and [2].

## 2. The Main Results

We begin by considering equation (1).

As $s(1) = 0 < 1 = \phi(1)$, we consider only the possible solutions $> 1$.

**Proposition 2.1.** *Of the positive integer solutions that belongs to one of the following classes:*

*(i) even integers*

*(ii) odd integers whose prime factorization contains a prime with odd power*

*(iii) integers of the form $p^{2\alpha}$ such that $p$ is an odd prime*

*(iv) integers of the form $p^{2\alpha}q^{2\beta}$ such that $p$ and $q$ are distinct odd primes*

*$n = 2$ is the only solution to equation (1).*

We will prove that Class (i) yields only one solution, namely $n = 2$, and the rest of the classes yields no solutions.

*Class (i):* Let $n = 2m$ for some positive integer $m$. Note that both 1 and $m$ divides $n$. If $m > 1$, then

$$s(n) \geq m + 1$$

On the other hand, half of the numbers between 1 and $n$ inclusively are even. Hence, at least half of them are not co-prime with $n$. So

$$\phi(n) \leq m$$

Therefore, for $n > 2$,

$$s(n) > \phi(n)$$

But note that

$$s(2) = 1 = \phi(2)$$

Therefore, if $n$ is even, then 2 is the only solution to equation (1), as desired.

*Class (ii):*

Let $n$ be an odd positive integer $> 1$ whose prime factorization is

$$p_1^{\alpha_1} \, p_2^{\alpha_2} \, ... \, p_k^{\alpha_k},$$

where $p_i$ are odd primes for all $i$, $1 \le i \le k$

We will prove that if there exists $j$ such that $\alpha_j$ is odd, then

$$s(n) \ne \phi(n)$$

Recall that

$$\phi(n) = p_1^{\alpha_1-1} p_2^{\alpha_2-1} ... \ p_k^{\alpha_k-1}(p_1 - 1)(p_2 - 1) \ ... \ (p_k - 1)$$

As $n$ is odd $> 1$, then all of its prime divisors are odd, including $p_1$. Therefore $(p_1 - 1)$ is even, which implies that $\phi(n)$ is even. Recall that

$$\sigma(n) = (1 + p_1 + p_1^2 + ... + p_1^{\alpha_1})(1 + p_2 + p_2^2 + ... + p_2^{\alpha_2}) ... (1 + p_k + p_k^2 + ... + p_k^{\alpha_k})$$

Note that if $\alpha_1$ is odd, then

$$(1 + p_1 + ... + p_1^{\alpha_1})$$

is even, as it is a sum of an even number of odd numbers.

This implies that $\sigma(n)$ is also even. But

$$s(n) = \sigma(n) - n$$

Hence, $s(n)$ is odd. Therefore, $s(n)$ and $\phi(n)$ have different parities, which implies that

$$s(n) \ne \phi(n)$$

as claimed.

*Class (iii):* Let $n = p^{2\alpha}$, where $p$ is an odd prime and $\alpha$ is a positive integer. Using the formulas of $\phi(n)$ and $s(n)$,

$$\phi(n) = p^{2\alpha} - p^{2\alpha-1} \text{ and } s(n) = \sigma(n) - n = 1 + p + ... + p^{2\alpha-1}$$

As $p \ge 3$,

$$p^{2\alpha} - p^{2\alpha-1} \ge 3p^{2\alpha-1} - p^{2\alpha-1} = 2p^{2\alpha-1}$$

Now, if $2p^{2\alpha-1} > 1 + p + ... + p^{2\alpha-1}$, then $\phi(n) > s(n)$ and we are done. Thus it suffices to show that

$$2p^{2\alpha-1} > 1 + p + ... + p^{2\alpha-1}$$

The above inequality can be rewritten as

$$p^{2\alpha-1} > 1 + p + \ldots + p^{2\alpha-2} = \frac{p^{2\alpha-1}-1}{p-1}$$

As $p - 1 > 1$ (because $p$ is odd prime),

$$\frac{p^{2\alpha-1}-1}{p-1} < p^{2\alpha-1} - 1 < p^{2\alpha-1},$$

as desired.

To prove our claim about Class (iv), we have to develop some machinery.

**Lemma I.** Let $n = p_1^{2\alpha_1} p_2^{2\alpha_2} \ldots p_k^{2\alpha_k}$, for distinct primes $p_i$, $i = 1, 2, \ldots, k$. If

$$\frac{p_1 p_2 \ldots p_k}{(p_1-1)(p_2-1)\ldots(p_k-1)} < \frac{1+\sqrt{5}}{2},$$

then $s(n) < \phi(n)$.

*Proof.* Consider the inequality

$$\frac{p_1^{2\alpha_1+1} p_2^{2\alpha_2+1} \ldots p_k^{2\alpha_k+1}}{(p_1-1)(p_2-1)\ldots(p_k-1)} - p_1^{2\alpha_1} p_2^{2\alpha_2} \ldots p_k^{2\alpha_k} < p_1^{2\alpha_1-1} p_2^{2\alpha_2-1} \ldots p_k^{2\alpha_k-1}(p_1-1)(p_2-1)\ldots(p_k-1)$$

which we will call inequality (*).

Dividing both sides of the inequality by the RHS yields

$$\frac{p_1^2 p_2^2 \ldots p_k^2}{(p_1-1)^2(p_2-1)^2 \ldots (p_k-1)^2} - \frac{p_1 p_2 \ldots p_k}{(p_1-1)(p_2-1)\ldots(p_k-1)} < 1$$

Let $x = \dfrac{p_1 p_2 \ldots p_k}{(p_1-1)(p_2-1)\ldots(p_k-1)}$. The inequality becomes

$$x^2 - x < 1$$

or

$$x^2 - x - 1 < 0$$

Solving this inequality yields

$$x \in \left( \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right)$$

or, equivalently,

$$\frac{p_1\,p_2\,...\,p_k}{(p_1-1)(p_2-1)\,...\,(p_k-1)} \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

As the $\dfrac{p_1\,p_2\,...\,p_k}{(p_1-1)(p_2-1)\,...\,(p_k-1)} > 0$,

then if $\dfrac{p_1\,p_2\,...\,p_k}{(p_1-1)(p_2-1)\,...\,(p_k-1)} < \dfrac{1+\sqrt{5}}{2}$, the inequality (*) holds.

But $s(n) =$

$$(1+p_1+p_1^2+...+p_1^{2\alpha_1})(1+p_2+p_2^2+...+p_2^{2\alpha_2})\,...\,(1+p_k+p_k^2+...+p_k^{2\alpha_k}) - p_1^{2\alpha_1}\,p_2^{2\alpha_2}\,...\,p_k^{2\alpha_k}$$

$$= \frac{(p_1^{2\alpha_1+1}-1)(p_2^{2\alpha_2+1}-1)\,...\,(p_k^{2\alpha_k+1}-1)}{(p_1-1)(p_2-1)\,...\,(p_k-1)} - p_1^{2\alpha_1}p_2^{2\alpha_2}\,...\,p_k^{2\alpha_k}$$

$$< \frac{p_1^{2\alpha_1+1}\,p_2^{2\alpha_2+1}\,...\,p_k^{2\alpha_k+1}}{(p_1-1)(p_2-1)\,...\,(p_k-1)} - p_1^{2\alpha_1}p_2^{2\alpha_2}\,...\,p_k^{2\alpha_k}$$

$$< p_1^{2\alpha_1-1}p_2^{2\alpha_2-1}\,...\,p_k^{2\alpha_k-1}(p_1-1)(p_2-1)\,...\,(p_k-1)$$

$= \phi(n)$, as claimed.

$\square$

**Corollary 2.2.** *Given any positive number $k > 1$, there exists a prime $P$ such that if $n$ is an odd prime with $k$ divisors and all the prime divisors of $n$ are greater than or equal to $P$ then $s(n) < \phi(n)$*

*Proof.* Denote by $p_n$ the $n^{th}$ prime number. Since the sequence $(p_n)$ increases to $+\infty$, the sequence $a_n = \dfrac{p_n}{p_n-1}$ is strictly decreasing and convergent to 1.

As $\left(\dfrac{1+\sqrt{5}}{2}\right)^{1/k} > 1$, one can find $p_{i_1} < p_{i_2} < ...\, p_{i_k}$ such that

$$\frac{p_{i_k}}{p_{i_k}-1} < ... < \frac{p_{i_2}}{p_{i_2}-1} < \frac{p_{i_1}}{p_{i_1}-1} < \left(\frac{1+\sqrt{5}}{2}\right)^{1/k}$$

Thus,

$$\left(\frac{p_{i_1}}{p_{i_1}-1}\right)\left(\frac{p_{i_2}}{p_{i_2}-1}\right)\,...\,\left(\frac{p_{i_k}}{p_{i_k}-1}\right) < \frac{1+\sqrt{5}}{2}$$

Let $P = p_{i_1}$ and $n = p_{i_1}^{2\alpha_{i_1}} p_{i_2}^{2\alpha_{i_2}} \dots p_{i_k}^{2\alpha_{i_k}}$. Then by Lemma I,

$$s(n) < \phi(n)$$

as desired. $\square$

We will need the following elementary inequality

$$\frac{n}{n-1} > \frac{m}{m-1}$$

for integers $m > n > 1$. Denote it by inequality (**).

Now we can finish the proof of Class (iv).

Let $n = p^{2\alpha}q^{2\beta}$ such that $p$ and $q$ are distinct odd primes with $p < q$. So

$$\phi(n) = p^{2\alpha-1}q^{2\beta-1}(p-1)(q-1)$$

and

$$s(n) = \sigma(n) - n = (1 + p + \dots + p^{2\alpha})(1 + q + \dots + q^{2\beta}) = \frac{(p^{2\alpha+1} - 1)(q^{2\beta+1} - 1)}{(p-1)(q-1)} - p^{2\alpha}q^{2\beta}$$

There are two cases:

Case I: $p \geq 5$

Then,

$$\frac{pq}{(p-1)(q-1)} < \frac{(5)(7)}{(4)(6)} = \frac{35}{24}$$

by inequality (**).

As $\dfrac{35}{24} < \dfrac{1 + \sqrt{5}}{2}$, then by Lemma I

$$s(n) < \phi(n)$$

Hence,

$$s(n) \neq \phi(n)$$

Case II: $p = 3$

if $q \geq 17$, then

$$\frac{pq}{(p-1)(q-1)} < \frac{(3)(17)}{(2)(16)} = \frac{51}{32}$$

again by inequality (**)

But $\dfrac{51}{32} < \dfrac{1 + \sqrt{5}}{2}$. Thus, by Lemma I, $s(n) < \phi(n)$

Upon checking the pairs $(p, q) = (3, 5), (3, 7), (3, 11), (3, 13)$, it turns out that none of them provides a solution to

$$s(n) = \phi(n)$$

By combining the two cases, it follows that no solution belongs to the Class (iv), as claimed.

After considering all of the four classes, we can conclude that if the equation

$$s(n) = \phi(n)$$

has a solution greater than 2, it must be an odd square such that the number of its distinct prime divisors is greater than 2 and $\displaystyle\prod_{p|n} \dfrac{p}{p-1} > \varphi$, where $p$ is prime and $\varphi$ is the famous golden ratio $= \dfrac{1 + \sqrt{5}}{2}$.

*Remark* 2.3. Lemma I provides a quick way to check if $s(n) < \phi(n)$ for square odd $n$, and, surprisingly, this method depends only on the prime factors, not on their powers (provided of course they are even).

We continue this section with equation (2).

**Proposition 2.4.** *The only solutions to the equation*

$$d(n) = \phi(n)$$

*are* $1, 3, 8, 10, 18, 24, 30$

The solution $n = 1$ is trivial, so assume that $n > 1$. We will prove some lemmas that will help us in proving the proposition. For the rest of the paper, let $\alpha$ be a positive integer, unless stated otherwise.

**Lemma II:** If $p$ is a prime $\geq 5$, then $\phi(p^\alpha) > d(p^\alpha)$ (i.e $p^{\alpha-1}(p-1) > \alpha+1$).

*Proof.* Note that $p^{\alpha-1}(p-1) \geq 5^{\alpha-1} \cdot 4$. We will prove by induction that $5^{\alpha-1} \cdot 4 > \alpha+1$

Base Case ($\alpha = 1$): $4 > 1 + 1$

Induction hypotheses: assume that $5^{k-1} \cdot 4 > k + 1$

So

$$5(5^{k-1} \cdot 4) > 5(k+1) > (k+1) + 1$$

Hence,

$$5^{k+1} \cdot 4 > (k+1) + 1$$

completing the induction.

We conclude that

$$p^{\alpha-1}(p-1) \geq 5^{\alpha-1} \cdot 4 > \alpha + 1 \text{ for all } \alpha \text{ and prime } p \geq 5, \text{ as desired.}$$

$\square$

**Lemma III:**

(i) $\phi(3^\alpha) = d(3^\alpha)$ if $\alpha = 1$ (i.e. $2 \cdot 3^0 = 2$ )

(ii) $\phi(3^\alpha) > d(3^\alpha)$ if $\alpha > 1$ (i.e. $2 \cdot 3^{\alpha-1} > \alpha + 1$ )

*Proof.* (i) is a direct substitution. We will prove (ii) by induction on $\alpha$ (starting from $\alpha = 2$).

Obviously, ($\alpha = 2$): $6 > 3$

Assume that $2 \cdot 3^{k-1} > k + 1$, for some positive integer $k \geq 2$. So

$$3(2 \cdot 3^{k-1}) > 3(k+1) > k + 2$$

Thus,

$$2 \cdot 3^k > k + 2$$

Induction is complete, and the lemma is proved. $\square$

**Lemma IV:**

(i) $\phi(2^\alpha) < d(2^\alpha)$ if $\alpha = 1$ or $2$

(ii) $\phi(2^\alpha) = d(2^\alpha)$ if $\alpha = 3$

(iii) $\phi(2^\alpha) > d(2^\alpha)$ if $\alpha > 3$ (i.e. $2^{\alpha-1} > \alpha + 1$ )

*Proof.* (i) and (ii) are direct substitution. As we did with Lemmas II and III, we will prove (iii) by induction on $\alpha$ (starting from $\alpha = 4$)

Base case ($\alpha = 2$): $8 > 5$, which is true.

Assume that $2^{k-1} > k + 1$, for some positive integer $k \geq 4$. So

$$2^k = 2(2^{k-1}) > 2(k+1) > k + 2$$

concluding the induction.                                                    □

**Lemma V:** For every $p > 3$, then $\phi(p^\alpha) \geq 2d(p^\alpha)$, with equality holds if and only if $p = 5$ and $\alpha = 1$.

*Proof.* We will prove by induction on $\alpha$ (starting from $\alpha = 2$) that $5^{\alpha-1} \cdot 4 > 2(\alpha + 1)$

Obviously, $5^1 \cdot 4 > 6$

Assume that $5^{k-1} \cdot 4 > 2(k+1)$, for some $k \geq 2$. Then

$$5(5^{k-1} \cdot 4) > 10(k+1) > 2(k+2),$$

finishing the induction.

As a result,

$$\phi(p^\alpha) = p^{\alpha-1}(p-1) \geq 5^{\alpha-1} \cdot 4 > 2(\alpha+1) = 2d(p^\alpha) \text{ for } k \geq 2.$$

Thus, the equality in Lemma V holds if and only if $p = 5$ and $\alpha = 1$.

□

**Lemma VI:** For any $\alpha \geq 2$, $\phi(3^\alpha) \geq 2d(3^\alpha)$, with equality holds if and only if $\alpha = 2$

*Proof.* Again, we will prove it by induction on $\alpha$ (starting from $\alpha = 3$) that

$$3^{\alpha-1} \cdot 2 > 2(\alpha + 1)$$

Base Case: $3^2 \cdot 2 > 8$

Assume that $3^{k-1} \cdot 2 > 2(k+1)$, for some $k \geq 3$

Then,

$$3(3^{k-1} \cdot 2) > 6(k+1) > 2(k+2)$$

So

$$3^k \cdot 2 > 2(k+2)$$

finishing the induction.

Therefore, for $\alpha \geq 3$.

$$\phi(3^\alpha) = 3^{\alpha-1} \cdot 2 > 2(\alpha+1) = 2d(3^\alpha)$$

Thus, the equality in the Lemma holds if and only if $\alpha \geq 2$, as claimed.          □

We will use the lemmas to prove the Proposition. Let $n = p_1^{\alpha_1} \ p_2^{\alpha_2} \ ... \ p_k^{\alpha_k}$, where $p_i$ are distinct primes for $1 \leq i \leq$ k, with $p_1$ is the smallest prime, and $\alpha_i$ is a positive integer for $1 \leq i \leq$ k. There are three cases to consider:

**First:** If $p_1 > 3$, no solution exists.

*Proof:* $\phi(n) =$

$\phi(p_1^{\alpha_1}) \ \phi(p_2^{\alpha_2}) \ ... \ \phi(p_k^{\alpha_k})$          (as $\phi(n)$ is a multiplicative function)

$> d(p_1^{\alpha_1}) \ d(p_2^{\alpha_2}) \ ... \ d(p_k^{\alpha_k})$          (Lemma II)

$= d(n)$          ($d(n)$ is multiplicative)

Hence, no solution exists, as claimed.

**Second:** If $p_1 = 3$, then $n = 3$ is the only solution.

*Proof:* There are two subcases:

Subcase I: the number of distinct primes that divides $n$ is greater than 1 (i.e $k \neq 1$)

By Lemma III: $\phi(3^{\alpha_1}) \geq d(3^{\alpha_1})$

By Lemma II: $\phi(p_2^{\alpha_2}) \ ... \ \phi(p_k^{\alpha_k}) > d(p_2^{\alpha_2}) \ ... \ d(p_k^{\alpha_k})$

Combining the previous statements, we conclude that

$$\phi(n) = \phi(3^{\alpha_1}) \ \phi(p_2^{\alpha_2}) \ ... \ \phi(p_k^{\alpha_k}) > d(3^{\alpha_1}) \ d(p_2^{\alpha_2}) \ ... \ d(p_k^{\alpha_k}) = d(n)$$

Therefore, no solution exists.

Subcase II: the number of distinct primes that divides $n$ is 1. Thus, $n = 3^{\alpha_1}$.

By Lemma III, $\alpha_1$ must be 1. $n = 3$ works, as claimed.

**Third:** If $p_1 = 2$, then there are four subcases to consider, based on the value of $\alpha_1$.

Subcase I: $\alpha_1 > 3$ Then,

$$\phi(n) = \phi(2^{\alpha_1})\, \phi(p_2^{\alpha_2}) \, ... \, \phi(p_k^{\alpha_k}) > d(2^{\alpha_1})\, d(p_2^{\alpha_2}) \, ... \, d(p_k^{\alpha_k}) = d(n)$$

by Lemmas II, III and IV. No solution exists.

Subcase II: $\alpha_1 = 3$.

If there is a prime number greater than 3 that divides $n$, then, again,

$$\phi(n) = \phi(2^{\alpha_1})\, \phi(p_2^{\alpha_2}) \, ... \, \phi(p_k^{\alpha_k}) > d(2^{\alpha_1})\, d(p_2^{\alpha_2}) \, ... \, d(p_k^{\alpha_k}) = d(n)$$

by Lemmas II, III and IV. No solution exists.

Therefore, we can limit our investigation to the cases $n = 2^3$ and $2^3 \cdot 3^{\alpha_2}$.

Note that $\phi(2^3) = d(2^3)$, so 8 is a solution. By Lemma III,

$$\phi(3^{\alpha_2}) \geq d(3^{\alpha_2})$$

It follows that

$$\phi(2^3 \cdot 3^{\alpha_2}) = \phi(2^3)\phi(3^{\alpha_2}) \geq d(2^3)d(3^{\alpha_2}) = d(2^3 \cdot 3^{\alpha_2})$$

by Lemmas III and IV. Equality holds if and only if $\alpha_2 = 1$. Indeed, $\phi(24) = d(24)$.
So 24 is a solution.

Subcase III: $\alpha_1 = 2$

Let $n = 2^2 \cdot m$, where $m$ is not divisible by 2. So

$$\phi(n) = \phi(2^2 \cdot m) = \phi(2^2)\phi(m) = 2\phi(m)$$

Similarly,

$$d(n) = d(2^2 \cdot m) = d(2^2)d(m) = 3d(m)$$

So our goal is to find the solutions of

$$2\phi(m) = 3d(m)$$

If $m$ is divisible by a prime $> 3$, let $m = p_2^{\alpha_2} \ p_3^{\alpha_3}...p_k^{\alpha_k}$ with, $p_j > 3$ for some $j$, $2 \leq j \leq k$ .

We claim that $2\phi(m) > 3d(m)$. Note that

$$\phi(m) = \phi(p_2^{\alpha_2}) \ ... \ \phi(p_j^{\alpha_j}) \ ... \ \phi(p_k^{\alpha_k})$$

$$\geq d(p_2^{\alpha_2}) \ ... \ 2d(p_3^{\alpha_3}) \ ... \ d(p_k^{\alpha_k}) \ (\text{Lemmas III and V })$$

$$> \frac{3}{2} \ d(p_2^{\alpha_2}) \ d(p_3^{\alpha_3}) \ ... \ d(p_k^{\alpha_k})$$

$$= \frac{3}{2} \ d(m)$$

Thus, $\phi(m) > \frac{3}{2} \ d(m)$, which is our claim.There$m$ must be equal to $3^\alpha$, for some positive integer $\alpha$.

Note that if $\alpha \geq 2$, then by Lemma VI,

$$\phi(m) \geq 2d(m) > \frac{3}{2} \ d(m)$$

Hence, $\alpha = 1$. However, $n = 2^2 \cdot 3$ does not satisfy the equation $\phi(n) = d(n)$. Thus, this subcase yields no solutions.

Subcase IV: $\alpha_1 = 1$.

Let $n = 2m$, where $m$ is odd. So

$$\phi(n) = \phi(2m) = \phi(2)\phi(m) = \phi(m)$$

Similarly,

$$d(n) = d(2m) = d(2)d(m) = 2d(m)$$

Our goal is to solve $\phi(m) = 2d(m)$.

Again, if $m$ is divisible a prime $> 3$, let $m = p_2^{\alpha_2} \ p_3^{\alpha_3}...p_k^{\alpha_k}$, with $p_j > 3$. Then by Lemmas III and V,

$$\phi(m) = \phi(p_2^{\alpha_2}) \ ... \ \phi(p_j^{\alpha_j}) \ ... \ \phi(p_k^{\alpha_k})$$

$$\geq d(p_2^{\alpha_2}) \ ... \ 2d(p_j^{\alpha_j}) \ ... \ d(p_k^{\alpha_k})$$

$$= 2d(m)$$

The equality holds only if the greatest prime that divides $m$ is 5. Otherwise, by Lemma V, the inequality will be strict. Also, the power of 5 must be 1 (Lemma V again).

This leaves us with two possibilities: $m = 3^{\alpha_2} \cdot 5$, for some positive integer $\alpha_2$, or $m = 5$.

Let m $= 3^{\alpha_2} \cdot 5$. Note that for $\alpha_2 \geq 2$,

$$\phi(m) = \phi(3^{\alpha_2} \cdot 5) = \phi(3^{\alpha_2}) \, \phi(5)$$

$$\geq 2 \cdot 2d(3^{\alpha_2}) \, d(5) \text{ (Lemma VI and } \phi(5) = 2d(5))$$

$$= 4d(3^{\alpha_2}) \, d(5) = 4d(3^{\alpha_2} \cdot 5) = 4d(m) > d(m).$$

Thus, we are left with only two possible solutions $m = 3^1 \cdot 5$ or $5$.

By checking these values, we know that they actually work, giving the solutions: $n = 2 \cdot 3 \cdot 5$ and $n = 2 \cdot 5$

Finally, we consider $m = 3^{\alpha_2}$, for some positive integer $\alpha_2$. If $\alpha_2 > 2$, by Lemma VI,

$$\phi(3^{\alpha_2}) > 2d(3^{\alpha_2})$$

So $m = 3$ or $3^2$. By checking, only $m = 3^2$ works. So $n = 2 \cdot 3^2$ is a solution.

Having considered all the possible cases, we can conclude that the only solutions to the equation $\phi(n) = d(n)$ are $n = 1, 3, 8, 10, 18, 24, 30$. (Q.E.D)

## ACKNOWLEDGEMENT

## References

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Wagner College, Staten Island, NY 10301, U.S.A.

*Email address*: mohamed.lotfi@wagner.edu