A SECOND COURSE IN ALGEBRA

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PREFACE

This book is the sequel to "A Year of Algebra," and is designed for use in the second or third year of the high-school course.

In common practice the mathematics courses in high schools require a year of geometry following the first year algebra, but this book does not assume a knowledge of geometry; it may follow "A Year of Algebra" in a continuous study of the subject through a year and a half or two years as required in some schools.

The first chapter of this book provides for a thorough review of the essentials taught in "A Year of Algebra." The extent to which this chapter will be used is left to the discretion of the teacher. Many of the examples should be worked orally during the first few days of the term. In this way the pupils revive their knowledge of algebraic facts and principles, and renew their habits of thinking in algebraic terms; while the teacher has an opportunity to study the preparation and power of his pupils and to decide where his teaching should begin.

In planning and organizing the material of this course the authors have drawn from a teaching experience of many years, following the general plan of "A Year of Algebra."

1. All new topics are introduced by carefully chosen, suggestive illustrative examples.

2. The problems are carefully graded, and, where teaching experience suggested the desirability, helpful hints and suggestions point the way, so that the student need never feel the discouragement of the proverbial blank wall.

3. As in "A Year of Algebra," the provisions for review are frequent and extensive. The few especially apt students may not need all of these, but those in the weaker half of any class need abundant review practice.

4. The lists of verbal problems, problems that are to be translated into equations and solved, are large and diversified. These challenge attention and foster lively interest throughout the course.

5. The knowledge and use of logarithms is presented as early as possible, and the application of logarithms is made attractive. All must recognize the great practical value of this work.

6. Both teacher and student will find that it is seldom necessary to work all of the examples in an exercise before the rules and principles are mastered.

—The Authors.

CONTENTS

Factor
H.C.F. - 245.
L.C.D.
Fractions.
Common D. + Solve 252.
Equadratic

SECOND COURSE IN ALGEBRA

CHAPTER XI

ESSENTIAL REVIEWS

132. Symbols. Brevity of expression is obtained in mathematics by the use of symbols. In general, such symbols may be classified according as they are used—

(1) to represent number or quantity,

(2) to indicate operations and establish relations.

In arithmetic, the digits of the so-called Arabic notation (more properly called the Hindoo notation, for the characters and the place system were probably invented by the Hindoos), 1, 2, . . . 9, stand for particular and definite numbers. Such numbers may be known as **arithmetical, Arabic,** or **Hindoo numbers.** Similarly in the old Roman notation the capital letters were used to represent particular numbers.

The simple formulas used in arithmetic, such as $A=lw$, $i=prt$, and $C=2\pi r$, introduced to the student the idea of representing general numbers by the use of the letters of the English and other alphabets. Numbers represented by letters are known as **literal numbers.**

It is often convenient to represent distinct numbers of the same general group by the use of subscripts, such as r_1 and r_2 (read "r sub 1 and r sub 2") for the radii of two circles.

Exercise. Let the student make a list of all the symbols of operation and relation that he has met so far in his study of mathematics, naming each and illustrating its use.

133. Definitions. An **algebraic expression,** or simply an **expression,** is any combination of symbols that represents a number or operations on numbers. The following are expressions: $2ab$, $a+b$, $x^3+3x^2y+3xy^2+y^3$, and $\frac{3a^2by}{7bxyz}$.

A **term** is an expression not separated by the signs + or − and representing a single number or quantity. The following are terms: $2ax$, $36x^2y$, $\frac{3a^2by}{7xyz}$, and $(a+b)$ which we read as "the quantity $a+b$."

Like or **similar terms** are terms that have the same literal parts, such as ab, $2ab$, and $7ab$, or $3a^2x$, $12a^2x$, and $77a^2x$, or $2(a+b)$, $5(a+b)$, and $25(a+b)$.

Define and write an illustration for each of the following: exponent, coefficient, monomial, binomial, trinomial, polynomial, equation of condition, identity.

In such a term as $21abxy$, 21 is called the **arithmetical** or **numerical coefficient** of the literal part, $abxy$. It should be recalled that ab is the coefficient of $21xy$ and that $21y$ is the coefficient of abx.

When we wish to represent the idea that two numbers are to be considered as opposite in character or direction, we make use of the signs + and −, and call such numbers **positive** and **negative numbers, directed numbers,** or + and − **numbers.** (See Chapter II.)

134. The following rules from Chapters II and III need emphasis.

Rule for the sign of the product of several factors. *The sign of a product is + if an even number of its factors are − and the sign of a product is − if an odd number of its factors are −.* (See §§ 30 and 39.)

Rule for the sign of a quotient. *If the dividend and divisor have like signs, the sign of the quotient is +. If they have unlike signs, the sign of the quotient is −.* (See § 31.)

Rules for the − sign with the signs of aggregation. *If an expression is placed within a sign of aggregation preceded by the − sign, the sign of each term of the expression must be changed.* (See § 37.)

If an expression is removed from a sign of aggregation preceded by the − sign, multiply each term by the coefficient before the sign of aggregation (if there is one) and change the sign of each term of the product. (See § 36.)

If an expression is placed within or removed from a sign of aggregation preceded by a + sign, no changes of signs are necessary.

Rule for exponents in multiplication. *The exponent of each literal number in a product of monomials is equal to the sum of the exponents of that number in the factors.*

It is convenient to remember this rule in its **typeforms** $a^m \cdot a^n \equiv a^{m+n}$ when a single literal number is involved and $(a^m b^n)(a^x b^y) \equiv a^{m+x} b^{n+y}$ when two or more numbers are involved. (See § 40.)

Rule for exponents in division. *The exponent of each literal number in a quotient of monomials is equal to its exponent in the dividend minus its exponent in the divisor.* (See § 41.)

The **typeforms** are $a^m \div a^n \equiv a^{m-n}$
and $a^m b^n \div a^x b^y \equiv a^{m-x} b^{n-y}$.

Evidently these rules apply when exponents are used with arithmetical numbers for $7^3 \cdot 7^2 = 7^5$ and $7^5 \div 7^3 = 7^2$.

Exercise 120

1. Write the sum, the difference, the product, and the quotient of the two literal numbers a and b; of $3x$ and $2y$; of abc and xyz.

2. Write the quotient when the product of x and y is divided by the sum of a and b.

3. If $a=2$, $b=3$, and $c=4$, find the numerical value of $3a^2b$; of a^3+b^2-c; of a^3+b^3; of $a^3+3a^2b+3ab^2+b^3$.

4. Give the rule for finding the sum of several similar algebraic terms, some positive and some negative.

5. Give the rule for finding the sum of several polynomials that have similar terms.

6. Give the rule for finding the difference of two polynomials that have similar terms.

7. What is the product of $(-a)(-b)(-c)$? What changes of signs will not change the sign of the product and why?

8. What is the product of $(-2)(+3)(-5)(-3)(+4)(-6)$?

9. Divide $-12a^3b^4c^5$ by $3ab^2c^3$ and state the rules used.

10. What is the product of a^2b, $-ab^2$, and a^3b^2?

11. If the quotient is b and the dividend is a, what is the divisor? What if the dividend were $-2a$?

12. If the product is x and the multiplicand y, what is the multiplier? If the product were $-2xy$?

13. Remove the parenthesis in each of the following and collect where possible: (1) $-2a-(a-b)$, (2) $a-b-(a+b-c)$, (3) $x+y-2z-(3x-2y+3z)$.

14. Express by an equation the fact that a is as much more than twice b as three times b is less than three times c.

15. Express by an equation the fact that x exceeds three times a by the quotient of y divided by twice c.

135. Fundamental processes with polynomials.

Exercise 121

1. Add $3x^3-5x^2+7x-3$, $8-x^2+x-x^3$, $7-2x^3$, $3x^2+4$, and x^3-2x^2+3x-4.

2. From the sum of $2x^2-xy+y^2$ and $x^2+3xy-y^2$, subtract the sum of $4x^2-y^2$ and $3x^2+xy+3y^2$.

3. Take the sum of $6-4a^3-a$ and $5a-1-3a^2$ from the sum of $2a^3+5-3a+2a^2$ and $3a^2-5a^3-4-a$.

4. Subtract $b-c+d-e$ from a.

5. Multiply $3x^2-5x-3$ by $2x^2+x-7$.

6. Multiply $2n^3-3n^2+4n-8$ by $3n^3+2n-3$.
7. Multiply $x^4+x^2y^2+y^4$ by $x^4-x^2y^2+y^4$.
8. Simplify $3n^2-2(n-3)(2n+1)$.
9. Simplify $16a^2-2(2a-3)^2-3(a-3)(2a+7)$.
10. Divide n^3-3n^2+3n-1 by n^2-2n+1.
11. Divide $4n^4-9n^2+30n-25$ by $2n^2-3n+5$.
12. Divide $x^2-y^2-2yz-z^2$ by $x-y-z$.
13. Divide $a^3+b^3+3abc-c^3$ by $a+b-c$.

136. Special products. If necessary, find the first product in each group of five by actual multiplication and use it as an example for obtaining the other four.

Exercise 122

I

1. $(a+b)^2$.
2. $(x+2y)^2$.
3. $(3x+5a)^2$.
4. $(2x^2+3y^2)^2$.
5. $(3a^2b^2+5x^2y^2)^2$.

II

1. $(a-b)^2$.
2. $(3x-7a)^2$.
3. $(2a^2-3ab)^2$.
4. $(x^m-y^n)^2$.
5. $(3x^a-5y^b)^2$.

III

1. $(a-b)(a+b)$.
2. $(3x-2y)(3x+2y)$.
3. $(5a^2-7b^3)(5a^2+7b^3)$.
4. $(x^m-y^n)(x^m+y^n)$.
5. $[(m-n)-7][(m-n)+7]$.

IV

1. $(a+2)(a+3)$.
2. $(x-5)(x-8)$.
3. $(x-3)(x+7)$.
4. $(x^2-3y)(x^2+7y)$.
5. $[(m-n)-8][(m-n)+3]$.

V

1. $(3a+2)(2a+3)$.
2. $(2x+3y)(3x+5y)$.
3. $(2a+3b)(3a-5b)$.
4. $(5m-3n)(2m+n)$.
5. $(x^2+7a)(3x^2-5a)$.

VI

1. $(a+b)(c+d)$.
2. $(m-n)(x+y)$.
3. $(x^2+y^2)(2a-3b)$.
4. $(3r+5s)(2a-7b)$.
5. $(x^2+y^2)(m^2-n^2)$.

VII

1. $(a+b+c)(a+b-c)$.
2. $(x+y-5)(x+y+5)$.
3. $(m-n-2a)(m-n+2a)$.
4. $(3x-5y+2z)(3x-5y-2z)$.
5. $(x^2+y^2-xy)(x^2+y^2+xy)$.

To here

VIII

1. $(a+b-c)(a-b+c)$.
2. $(x+y-z)(x-y+z)$.
3. $(2m-3n+4a)(2m+3n-4a)$.
4. $(3a-2b+5c)(3a+2b-5c)$.
5. $(x^2+y^2+z^2)(x^2-y^2-z^2)$.

137. Factoring. The following types of factoring have all been studied. (See §§ 56–61.)

Exercise 123

Reduce each expression to prime factors:

I. $21a^3b^2c^2-28a^4b^4c+35a^5b^4=7a^3b^2(3c^2-4ab^2c+5a^2b^2)$.
(A polynomial with a monomial factor.)

1. $12a^2b^3-16a^3b^2+28a^3b^3$.
2. $7m^2n^2-14m^3n+21mn^3$.
3. $17x^5y^5-34x^4y^4+51x^3y^3$.
4. $26a^7b^5-39a^5b^7+13a^5b^5$.
5. $11a^4b^5c^6+22a^5b^6c^7-33a^6b^7c^8+44a^7b^8c^9$.

II. $9m^2-12mn+4n^2=(3m-2n)(3m-2n)$.
(A trinomial that is the square of a binomial.)

Rule.

1. $n^2-8n+16$.
2. x^2+6x+9.
3. $16a^2-24ab+9b^2$.
4. $25m^2+40mn+16n^2$.
5. $x^4+2x^2y^2+y^4$.
6. $9x^6+12x^3y^3+4y^6$.

III. $m^2n^2-4=(mn-2)\ (mn+2)$.

(A binomial that is the difference of two squares.)

1. x^2-y^2.
2. $25a^2-36$.
3. $36m^2n^4-49x^4y^6$.
4. $(x-y)^2-z^2$.
5. $(2a+3b)^2-25$.

IV. $a^2-8ab+15b^2=(a-3b)\ (a-5b)$.

(A trinomial that is the product of two binomials with a common term.)

1. $n^2-10n+21$.
2. x^2-x-20.
3. $m^2-11mn+24n^2$.
4. $a^2+9ab+20b^2$.
5. $r^2-3r-40$.
6. $k^2+7k-60$.

V. $3x^2+5x+2=(x+1)\ (3x+2)$.

(A trinomial that is the product of two binomials with like terms.)

1. $6a^2-13a+6$.
2. $6x^2+5xy-6y^2$.
3. $7n^2-17n-12$.
4. $4r^2+8rs+3s^2$.
5. $12m^2n^2-25mn+12$.
6. $6k^6-37k^3+6$.

VI. $ax-nx-ay+ny=x(a-n)-y(a-n)=(x-y)(a-n)$.

(A polynomial that is the product of two polynomials with unlike terms.)

1. $2ax+3am-2bx-3bm$.
2. $a^2m^2+b^2m^2-a^2n-b^2n$.
3. $a^5-a^3x+a^2x^2-x^3$.
4. $6ax-4ay+9bx-6by$.
5. $ax-ay+bx-by+cx-cy$.

VII. $x^4-5x^2y^2+4y^4=(x^2-4y^2)\ (x^2-y^2)=$
$(x-2y)\ (x+2y)\ (x-y)\ (x+y)$.

(An expression that requires more than one process to reduce it to prime factors.) (See § 62.)

1. $7x^2-21xy+14y^2$.
2. a^3-9a^2+18a.
3. $m^4-2m^2n^2+n^4$.
4. a^4-b^4.
5. $81x^4-16$.
6. a^5-13a^3+36a.

7. $3x^4-x^2y^2-2y^4$.
10. $16m^4-8m^2n^2+n^4$.
8. $a^2m^2-a^2n^2-b^2m^2+b^2n^2$.
11. $4x^2-16x+16$.
9. $a^3-5a^2-4a+20$.
12. x^8-y^8.
13. $(a-b)^4-2(a-b)^2(x+y)^2+(x+y)^4$.
14. $32a^4-16a^2b^2+2b^4$.

written Lesson

Exercise 124. Miscellaneous Expressions

State under which of the preceding types each of the following may be factored, and find its factors:

perfect sq or any write out kind of factor

1. a^2-4y^2.
2. a^4-4a^2.
3. a^3b-ab^3.
4. $8x^3-18xy^2$.
5. $64-n^4$.
6. a^2-2a-8.
7. $6x^2-13xy+6y^2$.
8. $4a^2-15a-4$.
9. $3a^2-a-2$.
10. $3x^3-6x^2y+3xy^2$.
11. $a^2+2ab+b^2-9$.
12. $(m+n)^2-(a-b)^2$.
13. $(x-y)^2-(x-y)$.
14. $a^5-4a^4+4a^3$.
15. $5n^2-25n+30$.
16. $a^2-b^2-2bc-c^2$.
17. x^2-y^2+x-y. (Divide by $x-y$.)
18. x^3y-xy^3+x-y.
19. $a^2-b^2+2bc-c^2$.
20. $25-x^2+2xy-y^2$.
21. $16-(a-b)^2$.
22. $4a^2+11ay+6y^2$.
23. a^4-a^2-20.
24. $48a^4-3b^4$.
25. a^4-5a^2+4.
26. $a+b+a^2-b^2$.
27. $ax+bx+ay+by$.
28. $a^2+2ab+b^2-81$.
29. $9a^2-21a+10$.
30. $100-29a^2+a^4$.
31. $a^2b^2-2abcd+c^2d^2$.
32. $8m^2-11mn+3n^2$.
33. a^2+a-b^2+b.
34. $3x^2yz-6xy^2z^2+3y^3z^3$.
35. $9x^4-15x^2-36$.
36. $(a-b)^2-2(a-b)c+c^2$.
37. $(x+y)^2-(x+y)(a-b)-20(a-b)^2$.

Solve

138. Equations. State the fundamental axioms of the equation. (See § 19.)

Exercise 125

Solve the following equations and check:

1. $3x+8-5x-7=3+2x-22$.
2. $2(5n+1)+3(7-n)=-12$.

3. $8x-5(5x+3)+3=3(7-2x)$.

4. $7(2n-3)-6(2n+3)=8(3n-4)-3(7-2n)$.

Solve the following literal equations for x and check:

5. $3x(x+a)-(x+a)^2=2x^2$.

6. $(x+m)^2-(x-m)^2=16m^2$.

7. $a(x-a)-b(x-b)=(a-b)^2$.

Exercise 126. Quadratic Equations

Solve each of the following by factoring and check:

1. $x^2-x-6=0$.
2. $x^2-x=0$.
3. $a^3-5a^2+4a=0$.
4. $x^2-x=12$.
5. $9x^2-17x=2$.
6. $12a^2-25a=-12$.

Solve each of the following by completing the square or by the formula and check: (See §§ 124 and 125.)

7. $2x^2-5x+2=0$.
8. $4x^2-3x-1=0$.
9. $6x^2-13x+2=0$.
10. $3x^2-11x=4$.
11. $2(a-1)^2+2a^2=5a(a-1)$.
12. $(2n+1)^2-(n+1)^2=(3n-2)^2$.

139. Highest common factor and lowest common multiple. *The highest common factor of two or more expressions is the product of the factors that are common to all the expressions.*

The lowest common multiple of two or more expressions is the product of all the different factors of the expressions, each factor used as many times as it occurs in any one of the expressions.

Exercise 127

Find the H. C. F. of the following:

1. $12a^3b^2$, $16a^2b^3$, and $20a^2b^2$.
2. $25m^2n^3$, $100m^3n^2$, and $75m^3n^3$.
3. a^2-ab, $a^3-2a^2b+ab^2$, and $2a^3-3a^2b+ab^2$.
4. x^2-x-6, x^2-5x+6, $x^2-7x+12$, and $ax-3a$.
5. x^4-y^4, x^2-y^2, $x-y$, and $x^2-2xy+y^2$.
6. $3ab-3b$, $2an-2n$, and a^2-1.
7. x^2+6x+5, $x^2+3x-10$, and $x^2+2x-15$.

Find the L. C. M. of the following:

8. 36, 16, 10, and 35.
9. $6a^2b$, $12a^3b$, and $10a^3b^3$.
10. $ax+ay$, x^2-y^2, and $x^2+2xy+y^2$.
11. x^2-x-6, x^2+5x+6, and x^2-9.
12. $3a^2+5ab+2b^2$, $3a^2-5ab+2b^2$, and a^2-b^2.
13. n^2-3n+2, n^2-4n+3, and n^2-5n+6.

140. Fractions.

State what fundamental operations may be performed on the terms of a fraction without changing its value. (See § 68.)

Exercise 128

Reduce the following fractions to lower terms:

1. $\dfrac{6axy}{9xyz}$.　2. $\dfrac{-28a^3b^4c^7}{42a^4b^3c^6}$.　3. $\dfrac{-10x^7y^8z^9}{-40x^5y^8z^{11}}$.

4. $\dfrac{x^2-y^2}{ax-ay}$.　5. $\dfrac{a^2-4b^2}{a^2-3ab+2b^2}$.　6. $\dfrac{9x^3y^3-9x^2y^4}{12x^4y^2-12x^3y^3}$.

7. $\dfrac{n-m}{m^2-n^2}$.　8. $\dfrac{a^2-b^2}{b^2-2ab+a^2}$.　9. $\dfrac{3-x}{x^2-5x+6}$.

Exercise 129

Supply the missing terms in the following:

1. $\dfrac{2x}{3y}=\dfrac{}{12x^2y^2}$.　2. $\dfrac{a}{a-b}=\dfrac{}{a^2-b^2}$.

3. $\dfrac{5a}{1}=\dfrac{}{a-2b}$.　4. $\dfrac{a}{b-a}=\dfrac{}{a-b}$.

5. $\dfrac{1-n}{(1-2n)\ (1+2n)}=\dfrac{}{(2n-1)\ (2n+1)}$.

6. $\dfrac{x+3}{x-2}=\dfrac{}{x^2-5x+6}$.　7. $\dfrac{4ax}{-5by}=\dfrac{-12abxy}{}$.

8. $\dfrac{m-n}{m+n}=\dfrac{}{3am^2-3an^2}$.　9. $\dfrac{x-z}{x-y}=\dfrac{}{y^2-x^2}$.

Exercise 130

Change each of the following fractions to mixed expressions:

1. $\frac{12a^2-20a+3}{4a}=3a-5+\frac{3}{4a}$ Ans.

2. $\frac{3x^3-9x^2-12x-2}{3x}$.

3. $\frac{9a^2+b^2}{3a-b}$.

4. $\frac{8x^3+27}{2x-3}$.

Change each of the following mixed expressions to a fraction:

5. $x+3+\frac{3}{2x}=\frac{2x^2+6x+3}{2x}$ Ans.

6. $a+5+\frac{2a-3}{2a}$.

7. $a^2-ab+b^2-\frac{2b^3}{a+b}$.

8. $x-3-\frac{5x-9}{x+3}$.

9. $\frac{a+b}{a-b}+1$.

10. $\frac{x+y}{x-y}-1$.

Exercise 131

Perform at sight the following additions and subtractions:

1. $\frac{x}{y}-1$.

2. $\frac{a}{b}-\frac{b}{a}$.

3. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

4. $\frac{3}{x-y}+\frac{2}{x+y}$.

5. $\frac{x}{2}-\frac{2x}{3}+\frac{3x}{4}$.

6. $\frac{a-b}{a+b}+\frac{a+b}{a-b}$.

7. $\frac{m+n}{n}+\frac{n}{m-n}$.

8. $\frac{3}{x^2-9}-\frac{2}{x^2-5x+6}$.

9. $\frac{x}{y}-\frac{x}{x+y}-\frac{x^2}{y(x+y)}$.

Perform the following additions and subtractions:

10. $\frac{2}{a-1}-\frac{3}{1-a}=\frac{2}{a-1}+\frac{3}{a-1}=\frac{5}{a-1}$ Ans.

11. $\frac{3}{x-2}-\frac{2}{x+2}-\frac{2}{2-x}$.

12. $\frac{3}{x-y}+\frac{2}{x+y}-\frac{3y}{y^2-x^2}$.

Exercise 132

Perform the indicated operations in each of the following and reduce the result to its simplest form:

1. $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{7}{8}$.

2. $\frac{3}{4}\cdot\frac{5}{6}\div\frac{5}{8}$.

3. $\frac{6x^2y^3}{35a^3b^2}\cdot\frac{14a^2b^3}{27x^4y^4}$.

4. $\frac{36a^3b^2}{25x^2y}\cdot\frac{16xy}{27ab^3}\div\frac{8ax}{9by}.$

5. $\frac{n^2-n-6}{n^2-6n+9}\cdot\frac{n^2+n-6}{n^2-4}.$

6. $\frac{a-b}{a-c}\cdot\frac{b-c}{b-a}\div\frac{b-a}{a-c}.$

7. $\frac{x^2y^2-1}{x^2y^2-9}\cdot\frac{x^2y^2-xy-6}{x^2y^2-xy-2}.$

8. $\frac{x-y}{(a-b)\ (b-c)}\cdot\frac{x+y}{(a-c)\ (b-a)}\div\frac{x^2+y^2}{(c-a)\ (c-b)}.$

9. $\frac{a^2-2ab+b^2-c^2}{a^2+2ab+b^2-c^2}\cdot\frac{a^2-b^2-2bc-c^2}{a^2-b^2+2bc-c^2}.$

10. $\frac{5a^2-\ 7ab+2b^2}{12a^2-25ab+12b^2}\div\frac{10a^2+11ab-6b^2}{8a^2-\ 2ab-3b^2}.$

141. Fractional equations.

Exercise 133

Solve the following equations and check:

1. $\frac{x}{2}+\frac{x}{3}=10.$ 2. $\frac{x}{2}-\frac{x}{5}=6.$ 3. $\frac{y}{2}+\frac{y}{3}+\frac{y}{4}=-26.$

4. $\frac{n+3}{4}+\frac{n+1}{3}-\frac{n+8}{5}=6.$ 5. $\frac{2m+1}{3}+\frac{m-4}{2}=\frac{5m+1}{3}.$

6. $\frac{n-2}{(n-1)\ (n+1)}-\frac{1}{n-1}=\frac{3}{n+1}.$

7. $\frac{x-a}{3}+\frac{x+a}{5}=x-2a.$ (Solve for x.)

8. $\frac{x}{a+b}-\frac{x}{a-b}=\frac{1}{a+b}.$ (Solve for x.)

9. $\frac{1}{x-2a}+\frac{1}{x+a}=\frac{m+n}{x^2-ax-2a^2}.$ (Solve for x.)

142. Simultaneous systems of equations.

Exercise 134

Solve the following simultaneous linear systems and check:

(Follow the plan of eliminating one of the unknowns by addition or subtraction. Method I, § 87.)

1. $x+y=1$, $x-y=5.$ 2. $2x-y=6$, $3x+2y=2.$ 3. $2a+3b=1$, $3a-b=18.$

4. $5x+4y=8$
$7x-3y=-6.$

5. $3m-5n=4$
$5m+2n=-14.$

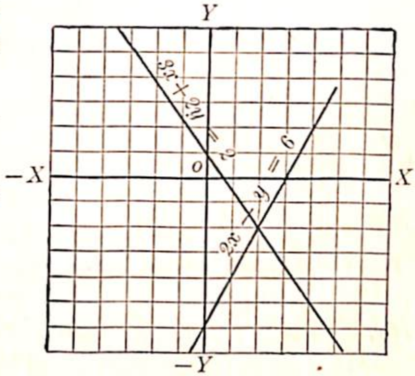
On the accompanying figure are the graphs of the equations in No. 2.

6. Solve No. 4 by graphing.

Solve the following for x and y:

7. $ax+by=2ab$
$ax-by=0.$

8. $mx+ny=k$
$ax+by=c.$



Exercise 135

Solve the following systems of equations, one linear and one quadratic, and check:

(**Hint.** Solve the linear equation for one unknown in terms of the other and substitute this value for that unknown in the quadratic.)

1. $x+y=5$
$x^2+y^2=25.$

2. $x-y=3$
$xy=10.$

3. $x^2-y^2=16$
$x-y=2.$

4. $x^2+y=12$
$x+y=0.$

5. $2x^2=y(x+6)$
$x+2y=7.$

6. $x^2+y^2-4x=21$
$x-y=1.$

Exercise 136. Problems

1. If the sum of two numbers is 15, and one is x, what is the other?

2. The sum of two numbers is 13. One-third the larger is one more than one-half the smaller. Find the numbers.

3. Write four consecutive numbers beginning with x. Write three consecutive odd numbers of which $x+1$ is the first.

4. The sum of three consecutive numbers is 33. What are they?

5. Find four consecutive even numbers whose sum is 60.

6. Find three consecutive even numbers such that the sum of the first two exceeds the third by 28.

7. Find four consecutive numbers such that the product of the third and fourth exceeds the product of the first and second by 26.

8. Find three consecutive integers whose sum exceeds half the largest by 37.

9. The second of three numbers is 3 less than twice the first and the third is 3 more than three times the first. The sum of the numbers is 42. What are they?

10. Separate 47 into two parts such that one part shall exceed the other by 15.

11. If $2x+9$ represents 37, what number will $x+5$ represent?

12. James has \$3.10 in dimes and quarters. If he has 3 more dimes than quarters, how many dimes has he?

13. The width of a rectangle is 3 inches more than half its length and its perimeter is 66 inches. Find its length and width.

14. The algebraic sum of three numbers is 7. The second is 11 less than the first and the third is 8 more than twice the second. Find the three numbers.

15. The average temperature of Thursday was 15° colder than that of Wednesday and the average temperature of Friday was 8° warmer than that of Thursday. Find the three temperatures if their sum was 11°.

16. John and James together have 40 cents. If twice the amount that John has be subtracted from 5 times the amount that James has the result is 25 cents. How much has each?

17. The first of the three angles of a triangle exceeds the second by 40° and the third angle equals half the sum of the other two. Find the angles.

(Hint. The sum of the angles of a triangle is 180°.)

18. A is twice as old as B and 20 years ago he was 4 times as old. Find their present ages.

19. The second of two numbers is 2 more than 3 times the first. If the second be subtracted from 5 times the first, the remainder is 12. Find the numbers.

20. Two-fifths of the sum of a certain number and 9 equals 20. What is the number?

21. Separate 48 into two parts such that one-half of one part added to one-third of the other part gives 22.

22. Separate 56 into two parts such that their quotient equals 4/3.

23. What number added to both the numerator and the denominator of 7/9 will make the resulting fraction equal to 9/10?

24. A sum of $600 was divided equally among a certain number of persons. If there had been 6 more persons, each would have received four-fifths as much. How many persons were there?

25. Find the time between 5 and 6 o'clock when the hands of the clock are together.

26. Find the time between 8 and 9 o'clock when the hands of a clock point in opposite directions.

27. Find the times between 5 and 6 o'clock when the hands of a clock are at right angles.

28. Find two numbers such that 3 times the first equals 5 times the second and the sum of 5 times the first and 2 times the second is 62.

29. I have $10.00 in dimes and quarters, 52 coins altogether. How many of each have I?

30. If 10 apples and 9 oranges cost 75 cents and at the same price 7 apples and 8 oranges cost 61 cents, find the cost of an apple and an orange.

31. There are three angles whose sum is 360°. The first is the supplement of one-third of the second and the first exceeds the third by 40°. Find the number of degrees in each angle.

32. A motor boat that can make 12 miles an hour in still water requires 6 hours to go 32 miles upstream and return. Find the rate of the stream.

33. Solve the formula $l=a+(n-1)d$ for n. Also for a and d.

34. Solve the formula $A=\frac{a(B+b)}{2}$ for b.

35. A man drove an automobile to a town 60 miles from his home and returned immediately by train. The whole trip required 4 hours and the average rate of the train was 16 miles per hour more than the average rate of the automobile. Find the two rates.

36. Find two numbers whose difference is 4 and the difference of whose squares is 34.

37. Find two numbers whose difference is 4 and the sum of whose squares is 40.

38. The hypotenuse of a right triangle is 2 inches longer than one side and 4 inches longer than the other. Find the three sides of the triangle.

Suggestion. Use the Pythagorean Proposition. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*

39. A rectangle is 4 inches longer than it is wide and its area is 45 sq. in. Find its dimensions.

40. Find the dimensions of a rectangle whose area is 84 sq. in. and whose length is 2 inches less than twice its width.

41. Find the dimensions of a rectangular field whose area is 240 square rods and whose perimeter is 64 rods.

42. Find the dimensions of a rectangular field whose area is 30 acres and whose perimeter is one mile.

CHAPTER XII

ADDITIONAL FUNDAMENTAL PROCESSES

143. Laws on the order of operations.

The sum of several numbers is the same in whatever order they are taken, for $a+b+c=a+c+b=b+c+a=c+b+a$.

This is known as the **commutative law for addition.**

It is obviously true for the sum of positive and negative numbers, for $2-3+4=2+(-3)+4$. Since subtraction is considered as the addition of a negative number, the law holds for subtraction. That is, $2-3=-3+2$.

The same law for multiplication states that *the factors of a product may be taken in any order*, for $2\cdot3\cdot4=2\cdot4\cdot3=4\cdot3\cdot2$.

Since division may be considered as the multiplication by a reciprocal and $4\div2\cdot3=4\cdot\frac{1}{2}\cdot3=4\cdot3\cdot\frac{1}{2}=3\cdot4\cdot\frac{1}{2}$, the same law applies in division.

Now $4+(3+5)=(4+3)+5=5+(3+4)$, and $b+(a+c)=(a+b)+c$. Evidently *the sum of several numbers is the same in whatever order they are grouped.*

This is known as the **associative law for addition.**

Also $4(5\cdot3)=(4\cdot5)3=5(4\cdot3)$ and $a(b\cdot c)=(a\cdot b)c=b(a\cdot c)$. *The product of several factors is the same in whatever order the factors may be grouped* is the same law for multiplication.

In arithmetical problems involving additions, subtractions, multiplications, and divisions, it is necessary to perform the multiplications and divisions first, then the additions and subtractions may be performed in any convenient order.

Illustrative example.

$$3-4\div2+5\cdot2\cdot6\div12=3-\frac{4}{2}+\frac{5\cdot2\cdot6}{12}=3-2+5=6.$$

Exercise 137

Simplify each of the following:

1. $4 \cdot 7 - 12 \cdot 2 \div 3 - 20 \div 5 \cdot 6$.

2. $27 - 9 \cdot 9 \div 3 + 4 \cdot 0 \cdot 8 \div 16$.

3. $12 + 12 \div 3 - 7 \cdot 8 \div 14 - 2(7 - 4 \div 2)$.

4. $10 - 5 \cdot 20 \div 25 + 18 \div 9 \cdot 3 \div 6 - \frac{1}{2}(5 - 7 \cdot 3 \cdot 2 \div 6)$.

5. $(12 - 8 \div 4 \cdot 3)\ (6 + 5 \div 10 - 2 \cdot 3 \div 4) - 7 \cdot 4 \cdot 3 \div 14$.

6. $\frac{1}{2} \div \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} \div \frac{1}{2} - \frac{7}{2} \cdot \frac{3}{4} \div 6 \div \frac{7}{3} + 2 \cdot 5 \cdot 3$.

The symbols of algebra are so definite that there is seldom a question as to the order of operations in evaluating an algebraic expression.

Evaluate each of the following, if $a = 2$, $b = 3$, $x = 1$, *and* $y = 4$:

7. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$.

8. $x^2y^2 - 2abxy + a^2b^2 + x^2 - y^2 + a^2 - b^2$.

9. $a^3b^3 - 3a^2b^2xy + 3abx^2y^2 - x^3y^3$.

144. The signs of aggregation. The common signs of aggregation are the **parenthesis** ( ), the **bracket** [ ], the **brace** { }, and the **vinculum** $\overline{\qquad\quad}$.

The vinculum has its most frequent use with the radical sign, $\sqrt{\quad}$. The bar of a fraction is a common sign of aggregation and is equivalent to a vinculum.

Whenever any one of these signs is used the expression enclosed is to be treated as a single quantity until the operation of removing the sign is performed. When simplifying expressions involving the use of more than one of the signs of aggregation, it is best to remove but one pair at a time beginning with the innermost.

Illustrative example.

$$2a - \{-3[a - (a - \overline{b - c}) - 2b] + c\} =$$
$$2a - \{-3[a - (a - b + c) - 2b] + c\} =$$
$$2a - \{-3[a - a + b - c - 2b] + c\} = 2a - \{-3[-b - c] + c\} =$$
$$2a - \{+3b + 3c + c\} = 2a - \{3b + 4c\} = 2a - 3b - 4c. \quad \text{Ans.}$$

Exercise 138

Remove all signs of aggregation and simplify each of the following:

1. $18-2(5-3)+3(-2[-2-\overline{4-2}-3])-2(-4+5)$.
2. $x-[x-(-\{-x-y\}-2y)-3y]+2(3[x+1])$.
3. $2x-2[-2(-2\{-2x+\overline{2y-2z}\}-4z)+8y]-16z$.
4. $x-(y-z-[x-\{y-z-\overline{x-y-z}-x\}-y]-z)-y$.
5. $2x-[3y-(4z-5w-\overline{x+y+z+w}-3y)-x]$.
6. $m+(m-n)-[m-(m-n)-\overline{m+n}]-(m+n)$.
7. $2x-2[-2(3x-2\overline{x-y})]=2x-2[-2(3x-2x+2y)]=?$

Note. The vinculum must be observed closely; $-2\overline{x-y}$ is the same as $-2(x-y)$, but is not $-\overline{2x-y}$.

8. $(-\overline{2a-5b})-(-2a-5b)$. Ans. $10b$.
9. $x-(2x-[3x-\{4x-\overline{5x-1}-2\}-1]-1)$.
10. $a-2\overline{b-3c}-\{-a+\overline{2b+3c}-2[a-b-3\overline{a+b-c}]-4a\}$.

Make the following expressions the difference of two squares by using signs of aggregation:

11. $a^2-b^2-c^2+2bc=a^2-(b^2-2bc+c^2)$.
12. $x^2-y^2-2ax-z^2+2yz+a^2$.
13. $9-x^2-y^2-6a-2xy+a^2$.
14. $12a-9-4a^2+m^2-2mn+n^2$.
15. $25x^2-20bx+4b^2-60ay-36y^2-25a^2$.
16. $4x^4-4x^2-1+x^6-4x^5+4x$.

In the following collect all negative terms and express as a single negative quantity:

17. $x^4-3x^3+6x^2-2x+1=x^4+6x^2+1-(3x^3+2x)$.
18. $x^2-ax-bx+cb=x^2-(ax+bx)+cb=x^2-(a+b)x+cb$.
19. $x^2-ax-bx-cx+abc$.
20. $x^3-ax^2-bx^2-cx^2+abx+acx+bcx+abc$.

145. Detached coefficients. When required to find the product or the quotient of two polynomials that can be arranged with reference to the same letter, the work can be shortened and the possibility of errors decreased by detaching coefficients. Note the following **illustrative examples:**

1. Multiply $3x^2-2x-7$ by $2x^2-3x-7$.

Old Process	Detaching coefficients	
$3x^2-2x-7$	$3-2-7$	$=-6$
$2x^2-3x-7$	$2-3-7$	$=-8$
$6x^4-4x^3-14x^2$	$6-4-14$	
$-9x^3+6x^2+21x$	$-9+6+21$	
$-21x^2+14x+49$	$-21+14+49$	
$6x^4-13x^3-29x^2+35x+49.$	$6-13-29+35+49$	$=48$

Supplying x gives $6x^4-13x^3-29x^2+35x+49$.

If 1 is substituted for x, notice the ease of checking for multiplicand $=-6$, multiplier $=-8$, and product $=48$.

Care must be taken to provide a 0 for each power of the literal number that is lacking in the arrangement.

2. Multiply $2x^4+3x^2-2x-4$ by x^3-3x^2+3 and check.

$$\begin{array}{rl}
2+0+3-2-4 & =-1 \\
1-3+0+3 & =+1 \\
\hline
2+0+3-2-4 & \\
-6+0-9+6+12 & \\
+6+0+9-6-12 & \\
\hline
2-6+3-5+2+21-6-12 & =-1
\end{array}$$

Supplying x gives $2x^7-6x^6+3x^5-5x^4+2x^3+21x^2-6x-12$.

3. Divide $15a^4-a+8a^2-1-19a^3$ by $5a^2-1-3a$.

Arranging and detaching coefficients gives

$$\begin{array}{r|l}
15-19+8-1-1 & 5-3-1 \\
15-9-3 & \overline{3-2+1} \\
-10+11-1 & 3a^2-2a+1. \text{ Ans.} \\
-10+6+2 & \\
\hline
+5-3-1 & \\
+5-3-1 & \\
\hline
\end{array}$$

On checking, the dividend $=2$, the divisor $=1$, and the quotient $=2$.

Exercise 139

Perform the indicated operation in each of the following, using detached coefficients:

1. Multiply a^3-3a^2+3a-1 by a^2-2a+1.

2. Multiply m^3-m^2+3m-5 by m^3+m^2+3m+5.

3. Divide $a^4-4a^3+6a^2-4a+1$ by a^2-2a+1.

4. Divide $m^4-3m^3-36m^2-71m-21$ by m^2-8m-3.

5. Multiply $1-7x^2+x^3+5x$ by $1+2x^2-4x$.

6. Multiply $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ by $a^2-2ab+b^2$.

7. Divide a^4+a^2+1 by a^2+a+1. Also by a^2-a+1.

8. Multiply x^5-2x^3+3 by $1-x^2+x$.

9. Divide $m^6-6m^4+5m^2-1$ by m^3+2m^2-m-1.

10. Divide $m^4+4m^2n^2+16n^4$ by $m^2+2mn+4n^2$.

11. Divide m^5-n^5 by $m-n$. Also by $m+n$.

12. Multiply $m^4-m^3n+m^2n^2-mn^3+n^4$ by $m+n$.

146. Synthetic division. When the divisor is a binomial of the type $x-a$ the work can be still further abbreviated.

Illustrative examples.

1. Divide x^3-5x^2+7x-2 by $x-2$.

I. Detached coefficients.

```
1-5+7-2|1-2
1-2    |1-3+1
  -3
  -3+6
    +1
    +1-2
```

II.

```
1-5+7-2|1-2
  -2   |1-3+1
  -3
    +6
    +1
      -2
```

Notice that II is obtained from I by omitting the first term of each partial product, for their subtraction is planned to eliminate the corresponding term of the dividend. Also notice that the first term of the dividend and of each partial dividend, 1, -3, $+1$, form the terms of the quotient in order and, since $+6$ is to be subtracted from $+7$, and -2 from -2, the form II may still further be compressed. See III and IV following. Notice that the quotient appears below.

III.

```
1-5+7-2|1-2
  -2+6-2|
1-3+1        Ans. x²-3x+1.
```

or

IV.

```
1-5+7-2|+2
  +2-6+2
1-3+1        Ans. x²-3x+1.
```

Now -2 is the essential term of the divisor and multiplication by it changes every sign of the partial products. But we are to subtract each of these partial products, therefore we may change -2 to $+2$ and add the partial products as in IV.

2. Divide x^3-2x^2-5x+8 by $x-3$.

Detaching coefficients and following IV gives:

```
1-2-5+8|+3
  +3+3-6
1+1-2. |2
```

Ans. $x^2+x-2+\dfrac{2}{x-3}$.

Exercise 140

1. Divide $a^3-3a^2-4a+12$ by $a-3$. Also by $a-2$.

2. Divide x^3+3x^2+x-2 by $x+2$. Also by $x+1$.

3. Divide x^3+2x^2-6x+8 by $x+4$.

4. Divide $a^3-22a+15$ by $a+5$.

5. Divide $x^4+3x^3-5x^2+4x-4$ by $x-2$.

6. Divide $m^4-m^3+5m^2-9m-10$ by $m-2$.

7. Divide $8x^3-24x^2+36x-27$ by $2x-3$.

Suggestion. Divide first by $x-3/2$, then divide the quotient by 2.

8. Divide $16a^4-81$ by $2a-3$. Also by $2a+3$.

9. Divide $6(x-y)^2-7(x-y)-20$ by $3(x-y)+4$.

10. Divide $3x^3-13x^2+23x-21$ by $3x-7$.

11. Check Numbers 7, 8, 9, and 10 by detaching coefficients and dividing as in § 145.

147. Additional special products and factoring.

Type VIII. The square of a polynomial. (For Types I to VII see § 136.)

$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$

and $(a-b-c)^2=a^2+b^2+c^2-2ab-2ac+2bc$.

Stated as a **rule** this becomes—

The square of a polynomial is equal to the sum of the squares of the terms together with twice the product of each term and every one that follows it.

Exercise 141

Expand each of the following:

1. $(m+n+p)^2$. **2.** $(x-y+z)^2$.

3. $(2x-3y-2z)^2$. **4.** $(3m+2n+3a)^2$.

5. $(x+y-m-n)^2$. **6.** $(x+w-3z+5)^2$.

The following are the squares of what polynomials?

7. $a^2+x^2+9y^2-2ax+6ay-6xy$.

8. $4m^2+25n^2+36p^2+20mn-24mp-60np$.

9. $a^2+4b^2+9c^2+49-4ab-6ac-14a+12bc+28b+42c$.

Type IX. The cube of a binomial.

Expand $(a+b)^3$. Expand $(a-b)^3$.

Exercise 142

Translate each of the formulas obtained by the expansions of $(a+b)^3$ and $(a-b)^3$ into a rule as in VIII and apply in the expansion of each of the following:

1. $(x+y)^3$. **2.** $(2-a)^3$. **3.** $(2x+y)^3$.

4. $(3m-2n)^3$. **5.** $(2ab+c)^3$. **6.** $(x^2-3y^2)^3$.

What are the factors of the following?

7. $27-27x+9x^2-x^3$.

8. $64-240a+300a^2-125a^3$.

9. $8a^3-36a^2b+54ab^2-27b^3$.

10. $a^6-9a^4x+27a^2x^2-27x^3$.

Type X. The factors of the sum or the difference of two cubes. Find by multiplication the products

$(a+b)\ (a^2-ab+b^2)$ and $(a-b)\ (a^2+ab+b^2)$.

Translate into a rule reading:—"*The factors of the sum of two cubes are the sum of the cube roots . . . ,*" and "*the factors of the difference of two cubes are . . . .*"

Exercise 143

Write at sight the following products:

1. $(x+y)\ (x^2-xy+y^2)$. **2.** $(x-y)\ (x^2+xy+y^2)$.

3. $(m+2)\ (m^2-2m+4)$. **4.** $(2a-b)\ (4a^2+2ab+b^2)$.

5. $(x^2+y^2)\ (x^4-x^2y^2+y^4)$.

6. $(2a^2-3b)\ (4a^4+6a^2b+9b^2)$.

Write at sight the factors for each of the following:

7. m^3+n^3. **8.** m^3-n^3. **9.** x^3+1.

10. x^3-1. **11.** a^3+8. **12.** a^3+8b^3.

13. $27m^3-1$. **14.** $8n^3+27r^3$. **15.** $m^3n^3+y^3$.

16. $125+y^3$. **17.** $a^3b^3-x^3y^3$. **18.** c^3+343.

19. a^6+1. Treat as $(a^2)^3+1$. **20.** $125-m^6$.

21. $8m^6-27n^6$. **22.** $64a^9-x^3$. **23.** $x^{12}+1$.

24. x^6-64. **25.** x^9-y^6. **26.** $(x-y)^3-8$.

Type XI. Find the product of a^2+ab+b^2 and a^2-ab+b^2 in two ways:

(1) by detaching coefficients,

(2) by treating the trinomials as the sum and the difference of two terms, i.e.,

$[(a^2+b^2)+ab]\,[(a^2+b^2)-ab]$.

Exercise 144

Write the product for each of the following:

1. $(x^2+xy+y^2)\,(x^2-xy+y^2)$.

2. $(n^2+3mn+9m^2)\,(n^2-3mn+9m^2)$.

3. $(4a^2-6ab+9b^2)\,(4a^2+6ab+9b^2)$.

4. $(x^4-x^2y^2+y^4)\,(x^4+x^2y^2+y^4)$

5. $(16a^2-4a+1)\,(16a^2+4a+1)$.

Study the products obtained in Numbers 1–5, then reverse the process and factor each of the following, using the plan of Number 6:

6. $a^4+a^2b^2+b^4=a^4+2a^2b^2+b^4-a^2b^2$
$=(a^2+b^2)^2-a^2b^2=[a^2+b^2+ab]\,[a^2+b^2-ab]$.

7. x^4+x^2+1. **8.** $1+a^2+a^4$. **9.** $x^8+x^4y^4+y^8$.

10. $a^4+3a^2+4=a^4+4a^2+4-a^2=?$

11. a^4-3a^2+9. **12.** $a^4+a^2b^2+25b^4$.

13. $x^4-7x^2+1=x^4+2x^2+1-9x^2=?$

14. $4a^4+7a^2b^2+4b^4$.

15. $x^4+4=x^4+4x^2+4-4x^2=?$

16. $4x^4+1$. **17.** $64+x^4$. **18.** $64x^4+1$.

19. $x^4+\frac{1}{4}$. **20.** x^4-3x^2+1.

21. x^4-11x^2+1. **22.** x^4-18x^2+1.

23. Make and factor three others of the type of 20, 21, and 22.

24. $x^4-14x^2+1=x^4+2x^2+1-16x^2=?$

25. x^4-23x^2+1. **26.** x^4-34x^2+1.

27. Make and factor three others of the type of 24, 25, and 26.

148. The Remainder Theorem and the Factor Theorem.

A **rational number** is one that can be expressed as a single whole number or as the quotient of two whole numbers.

If any expression of the form x^3+2x^2+3x+2 is divided by a binomial of the form $x-2$, the remainder is the same as if $+2$ were substituted for x.

Show by synthetic division that $(x^3+2x^2+3x+2)\div(x-2)$ gives $x^2+4x+11$ and remainder $+24$. Also substituting 2 for x, it gives $(2)^3+2(2)^2+3(2)+2=24$.

In general, $(x^2+ax+b)\div(x-n)$ gives $x+a+n$ and remainder n^2+an+b (check by division), therefore we have the **Remainder Theorem:**

If any rational integral expression in x is divided by $x-n$, the remainder is the same as if n were substituted for x.

Evidently, when the remainder becomes 0, $x-n$ is a factor and we have the **Factor Theorem:**

If any rational integral expression in x becomes 0 when n is substituted for x, then $x-n$ is a factor of the expression.

This theorem is useful in factoring expressions of a degree higher than the second.

Illustrative examples.

1. Factor x^3+3x^2+3x+2. Evidently, if $x-n$ is a factor of this expression, then n must be a factor of 2. Now the factors of 2 are $+1$, -1, $+2$, and -2. If we substitute $+1$ for x, the expression becomes $1+3+3+2$, or 9. If we substitute -1, it becomes $-1+3-3+2$, or $+1$. Similarly $+2$ gives $+28$, but -2 gives 0, therefore $x-(-2)$ or $x+2$ is a factor. By synthetic division, the quotient is x^2+x+1 and the remainder is 0.

2. Factor x^3-2x^2-5x+6. The factors of 6 are $+1$, -1, $+2$, -2, $+3$, -3, $+6$, and -6. Substituting each of these in turn for x, we find that $+1$, -2, and $+3$ each makes the expression 0. Therefore $x-1$, $x+2$, and $x-3$ are the required factors. (Check by multiplication.)

3. Factor $x^3-ax^2-14a^2x+24a^3$. Some of the factors of $24a^3$ are $+a$, $-a$, $+2a$, $-2a$, $+3a$, $-3a$, $+4a$, $-4a$, ... Of these $+2a$, $+3a$, and $-4a$ makes the expression 0. Therefore $x-2a$, $x-3a$, and $x+4a$ are the required factors.

Exercise 145

Factor:

1. x^3-7x^2+6.
2. x^3-7x-6.
3. a^3-10a^2+9.
4. $a^3-20a+32$.
5. x^3-4x^2+5x-2.
6. x^3-6x^2+5x+6.
7. x^3+3x^2-6x-8.
8. $a^3+12a^2+12a-45$.
9. m^4-3m^3+5m-2.
10. a^3-5a^2+4a+4.
11. $a^4-3a^3-a^2+a+6$.
12. $m^5-3m^2+24m-68$.
13. $m^3-3m^2n+4mn^2-4n^3$. (Try $m=+n$, $-n$, $+2n$, etc.)
14. $a^3-2a^2n-5an^2+6n^3$.
15. $a^3-5a^2y+18y^3$.

149. Typeform $a^n \pm b^n$. (The sum or difference of two like powers.)

We have learned how to factor a number of expressions that come under this typeform such as x^2-y^2, x^3-y^3, x^3+y^3 . . . and have discovered that $x+y$ and $x-y$ are included among the factors.

The factor theorem makes clear the rule for determining when either or both are divisors. If $+b$ is substituted for a in a^n-b^n, the expression becomes $(+b)^n-b^n$ which equals 0. Therefore $a-b$ is a factor of a^n-b^n for any integral value of n.

If $-b$ is substituted for a, a^n-b^n becomes $(-b)^n-b^n$, or $+b^n-b^n$ when n is even and it becomes $-b^n-b^n$, or $-2b^n$, when n is odd. Therefore $a+b$ is a factor of a^n-b^n when n is even but not when n is odd.

Now if $+b$ is substituted for a in a^n+b^n, the expression becomes b^n+b^n, or $2b^n$, no matter whether n is odd or even. Therefore $a-b$ is never a divisor of a^n+b^n. But if $-b$ is substituted for a, a^n+b^n becomes $(-b)^n+b^n$ which equals 0 when n is odd and $2b^n$ when n is even. Therefore $a+b$ is a divisor of a^n+b^n when n is odd. This gives the **rule:**

The expression a^n-b^n with n an integer is always divisible by $a-b$. It is divisible by $a+b$ when n is even.

The expression a^n+b^n is never divisible by $a-b$. It is divisible by $a+b$ when n is odd.

Exercise 146

1. Divide x^5-y^5 by $x-y$ using synthetic division.

Solution.

$$\begin{array}{l|l} 1+0+0+0+0-1 & +1 \\ \quad\ \ 1+1+1+1+1 & \\ \hline 1+1+1+1+1 & \end{array}$$

Ans. $x^4+x^3y+x^2y^2+xy^3+y^4$.

2. Divide x^5+y^5 by $x+y$ following the plan of Ex. 1.
3. Write the quotient of $(x^7-y^7)\div(x-y)$.
4. Write the quotient of $(x^7+y^7)\div(x+y)$.

Factor each of the following:

5. x^5-32. (Treat as x^5-2^5.)
6. x^5+243.
7. a^6+b^6. [Treat as $(a^2)^3+(b^2)^3$.]
8. a^9+b^9.
9. a^6+b^9. [Treat as $(a^2)^3+(b^3)^3$.]
10. a^6+125.

Exercise 147. Miscellaneous Exercises

Factor each of the following:

1. x^6-64.
2. x^6+64.
3. x^9+64.
4. $x^{12}+y^{12}$.
5. $x^{12}-y^{12}$.
6. $a^4+2a^3y-2ay^3-y^4$.
7. $x^2y^2+17xy+16$.
8. y^2-x^2-2x-1.
9. $x^2y^2+25-9z^2-10xy$.
10. x^8-34x^4+1.
11. x^2-y^2+x+y.
12. x^3-x^2+3x+5.
13. x^4+2x^2+9.
14. x^4-3x^2+9.
15. x^6+125y^3.
16. x^3-15x^2+250.
17. $10x^4-47x^2+42$.
18. a^6-64b^3.
19. $75a^2b^2-108c^2d^2$.
20. x^3-y^3+x-y.
21. x^3+x^2+x+1.
22. x^8+4.
23. $2x^2+3x-2$.
24. a^3-a^2-5a+2.
25. $36r^4-21r^2+1$.
26. $125m^3-150m^2+45m-2$.
27. $10x^2+3x-18$.
28. $abx^3+x+ab+1$.
29. $m^2-n^2+m^3-n^3$.
30. $a^2(a^2-1)-b^2(b^2-1)$.
31. $aby^3+y+ab+1$.
32. $7a^3x^2+49a^2x+84a$.
33. $(m-n)^2-9(m-n)-36$.
34. a^7+b^7.
35. $2a^3+7a^2+4a-4$.
36. y^4+y^3-3-3y.
37. $3x^6+8x^4-8x^2-3$.
38. $(a^3+1)^3-(b^3-1)^3$.
39. x^9-y^9.

150. Fractions.

Since a fraction is an indicated division it has three signs: (1) the sign of its numerator, or the dividend, (2) the sign of its denominator, or the divisor, (3) the sign of the value of the fraction, or the quotient. This is placed before the fraction and on a line with its bar. (See § 69.)

Any two of the signs of a fraction may be changed without changing the value of the fraction. The following are equivalent fractions: $\frac{a}{b}$, $\frac{-a}{-b}$, $-\frac{-a}{b}$, and $-\frac{a}{-b}$.

If $a=10$ and $b=2$ the value of each fraction is $+5$.

Similarly $\frac{a}{b-c}=\frac{-a}{c-b}=-\frac{a}{c-b}=-\frac{-a}{b-c}$.

Check by letting $a=6$, $b=4$, and $c=2$.

It will be recalled that the sign of the product of several positive and negative factors is + if there is an even number of negative factors and − if there is an odd number of negative factors. (See § 39.) This leads to the following:

Rule. *If the signs of an even number of factors are changed, the sign of their product is not changed. If the signs of an odd number of factors are changed, the sign of their product is changed.*

Exercise 148

1. Show that $(x-1)(x-2)=(2-x)(1-x)=-(x-1)(2-x)$ $=-(1-x)(x-2)$ by finding each of the four products by multiplication.

2. Write the product $(x-1)(x-2)(x-3)$ in as many different ways as possible.

Hint. $(x-1)$ $2-x)$ $(3-x)$.
Check by letting $x=5$ in each.

3. Write the fraction $\frac{x}{2b-a}$ in several equivalent forms. Check by letting $a=5$, $b=4$, and $x=6$.

4. Write the fraction $\frac{a-b}{c-d}$ in three other equivalent forms and check with $a=5$, $b=3$, $c=2$, and $d=1$.

5. Write the fraction $\frac{b-c}{(a-b)(a-c)}$ in three other equivalent forms and without changing the sign of the value of the fraction check in each.

6. The fraction $\frac{a-b}{(a-c)(b-c)(a-d)}$ may be written in seven other equivalent forms without changing the sign of the value of the fraction. Write these forms.

151. An important assumption of fractions is the following:

Axiom. *The value of a fraction is not changed if both numerator and denominator are multiplied or divided by the same quantity.* (See Axiom VI, § 68.)

Under this axiom we have simplified fractions, or reduced them to their lowest terms, by removing all common factors from the numerator and denominator.

Exercise 149

Reduce each of the following fractions to its lowest terms:

1. $\frac{x-y}{x^3-y^3}$.
2. $\frac{x^2-4}{x^3-8}$.
3. $\frac{a^2-a-2}{a^3-3a^2+3a-2}$.
4. $\frac{x^3-y^3}{x^5-y^5}$.
5. $\frac{8x^3+1}{4x^4-17x^2+4}$.
6. $\frac{x^4-y^4}{x^6+y^6}$.
7. $\frac{x^3-3x^2+4}{x^4-10x+4}$.
8. $\frac{b^2-2bc+c^2-d^2}{d^2-2cd+c^2-b^2}$.
9. $\frac{x^6+y^6}{x^{10}+y^{10}}$.
10. $\frac{a^2-4b^2+8bc-4c^2}{a^3+8(b-c)^3}$.

answer

152. In arithmetic, what is a proper fraction? Give an illustration. What is an improper fraction? Give an illustration.

An **improper algebraic fraction** is one the numerator of which contains a power of a literal number equal to or higher than the highest power of the same number in the denominator.

How would you define a proper algebraic fraction?

Exercise 150

Give the rule for changing a mixed expression to an equivalent improper fraction. Change each of the following mixed expressions into its equivalent improper fraction:

1. $x-y+\frac{x^3+y^3}{x^2+xy+y^2}$.
2. $x^2-x-2+\frac{2}{x^2+x-2}$.
3. $x^2-xy+y^2-\frac{x^4+y^4}{x^2+xy+y^2}$.
4. $x^4-x^3+x^2-x+1+\frac{2}{x+1}$.

Give the rule for changing an improper fraction into its equivalent mixed expression. Change each of the following fractions into mixed expressions:

5. $\frac{a^5-b^5}{a+b}$.
6. $\frac{x^3-3x^2-5}{x^2-x-1}$.
7. $\frac{x^5}{x^2+x+1}$.
8. $\frac{a^4+a^2b^2+b^4+3}{a^2+ab+b^2}$.
9. $\frac{a^5+c^5}{a-c}$.

153. Addition and subtraction of fractions.

Give the rule for finding the algebraic sum of several fractions with different denominators.

Exercise 151

Combine the following:

1. $\frac{x+1}{x^2-5x+6}+\frac{x+2}{x^2-7x+12}+\frac{x+3}{x^2-6x+8}$.
2. $\frac{1}{(a-b)\ (b-c)}-\frac{2}{(b-a)\ (b-c)}$. Ans. $\frac{3}{(a-b)\ (b-c)}$.
3. $\frac{2}{x+1}-\frac{3}{x-1}+\frac{1}{1-x^2}$.
4. $\frac{a+1}{(a-4)\ (c-2)}+\frac{a+4}{(a-1)\ (2-c)}$.

5. $\frac{m}{(m-n)\ (m-p)}+\frac{n}{(n-p)\ (n-m)}+\frac{p}{(p-m)\ (p-n)}.$

6. $\frac{x-3a}{x^2-3ax+9a^2}+\frac{3ax-2x^2}{x^3+27a^3}+\frac{1}{3a+x}.$

7. $\frac{a-1}{a^2-a+1}+\frac{2}{a^4+a^2+1}+\frac{a+1}{a^2+a+1}.$

8. $\frac{2}{a+3}-\left\{3-\frac{1}{a-3}-\left[\frac{1}{a^2-9}+\frac{a}{a-3}\right]\right\}.$

9. $\frac{(x-y)\ (z-y)}{y(y-a)}+\frac{xz}{ay}+\frac{(x-a)\ (z-a)}{a(a-y)}.$

154. Multiplication and division of fractions.

Exercise 152

Perform the indicated operations in each of the following and reduce to lowest terms:

1. $\frac{a^2-b^2}{b^3-a^3}\cdot\frac{a^4+a^2b^2+b^4}{b^3+a^3}.$ Ans. -1.

2. $\frac{ab}{a+b}\cdot\left(\frac{a}{b}-\frac{b}{a}\right)\cdot\frac{1}{a-b}.$

3. $\left(\frac{xy}{x+y}\right)\left(\frac{x+y}{x}+\frac{y}{x+y}\right).$

4. $\left(a+2b+\frac{b^2}{a}\right)\left(a-2b+\frac{b^2}{a}\right)\cdot\frac{a^2}{a^2-b^2}.$

5. $\left(x-\frac{y^2}{x}\right)\div\left(\frac{y^2}{x}+2y+x\right)\cdot\frac{x+y}{x-y}.$ Ans. 1.

6. $\left(a-b-\frac{a^2+b^2}{a+b}\right)\left(a+b-\frac{a^2+b^2}{a-b}\right).$

7. $\left(\frac{x}{a+x}-\frac{a-x}{x}\right)\div\left(\frac{x}{x+a}+\frac{a-x}{x}\right).$

8. $\frac{(a+b)^2-x^2}{2a+2b-2x}\div\frac{(b+x)^2-a^2}{a+b+x}\cdot\left(\frac{a-x+b}{a+b+x}\right)^2.$

9. $\dfrac{\frac{1}{1-a}+\frac{1}{1+a}}{\frac{1}{1+a}-\frac{1}{1-a}}=\dfrac{\frac{2}{1-a^2}}{\frac{-2a}{1-a^2}}=?$

10. $\dfrac{\frac{x-y}{y}+\frac{y}{x+y}}{\frac{1}{x}+\frac{1}{y}}.$

155. Equations. The statement by the use of the symbol of equality (=) that two number expressions are equal is called an **equation.** The number expressions are known as the **left,** or **first member,** and the **right,** or **second member,** according to their position with reference to the equality sign.

There are two classes of equations: (1) **identical equations,** or simply **identities,** (2) **equations of condition, conditional equations,** or simply **equations.**

An **identity** gets its name from the fact that its two members are either identically the same or become the same when the indicated operations are performed. The sign $\equiv$, read "is identically equal to," might be conveniently used in place of the sign = in identities. We have used it in the typeforms. An identity is true for every possible value of the literal numbers involved and is, therefore, of little use in the solution of problems.

The statements: (1) $a+2b=2b+a$, (2) $5=5$, and (3) $(x+y)(x-y)=(x^2-y^2)$ are identities. The members of (1) and (2) are exactly the same, and the first member of (3) becomes exactly the same as the second when the indicated multiplication is performed.

An **equation of condition** is an equation involving one or more unknowns which is a true equality only for certain values of these unknowns. The equation $2x-3=7$ is true only when x is 5. The equation $x^2-x=6$ is true only when x is 3 or -2. The equation $x+y=7$ is true when x is 3 and y is 4, when x is 5 and y is 2, or when x is 9 and y is -2, but would not be true if x were 6 and y were 3, or if x were 3 and y were 2.

The four **axioms** of the equation are:

(1) *The same number may be added to both members of an equation without destroying the equality.*

(2) *The same number may be subtracted from both members of an equation without destroying the equality.*

(3) *Each member of an equation may be multiplied by the same number without destroying the equality.*

(4) *Each member of an equation may be divided by the same number (not zero) without destroying the equality.*

How do these four axioms compare with the corresponding axioms of Geometry?

156. Fractional equations.

Exercise 153

Solve the following equations and check:

1. $\frac{3}{5x}+\frac{1}{2x}=\frac{11}{20}$.
2. $\frac{2}{x+3}=\frac{1}{x-2}$.
3. $\frac{x+1}{x-2}=\frac{x-3}{x-5}$.
4. $\frac{4x-3}{2x-1}=\frac{4x-5}{2x-7}$.
5. $\frac{12x-5}{21}=\frac{3x+4}{9x+3}+\frac{4x-5}{7}$.

Hint. Multiplying first by 21 and collecting gives $10=\frac{21x+28}{3x+1}$.

6. $\frac{1}{2}+\frac{2}{x+2}=\frac{13}{8}-\frac{5x}{4x+8}$.
7. $\frac{3x+1}{x-5}=\frac{5x+4}{x+8}-2$.
8. $\frac{x-7}{x+7}-\frac{2x-15}{2x-6}=-\frac{1}{2(x+7)}$.
9. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}$.
10. $\frac{2x+19}{5x^2-5}-\frac{17}{x^2-1}=\frac{3}{1-x}$.
11. $\frac{x+1}{(x+2)\ (x+3)}+\frac{x+2}{(x+1)\ (x+3)}=\frac{x+3}{(x+1)\ (x+2)}$.

To here

Hint. Multiplying by the L. C. D. $(x+1)\ (x+2)\ (x+3)$ gives $(x+1)^2+(x+2)^2=(x+3)^2$

Whence $x^2=4$, $\therefore x=\pm 2$, but -2 does not check because we cannot interpret $\frac{-1}{0}+\frac{0}{-1}=\frac{1}{0}$.

In general, no number can be accepted as a root of an equation if, when it is substituted for the unknown, any denominator reduces to zero. If the numerator of a fraction alone reduces to zero, the value of the fraction is assumed to become zero.

See Chapter XXII for further discussion of Indeterminate Forms.

12. $\dfrac{x-1}{(x-2)\,(x-3)}-\dfrac{x-2}{(x-1)\,(x-3)}=\dfrac{x-3}{(x-1)\,(x-2)}.$

Solution. Clearing of fractions and collecting gives $x^2-8x+12=0$ $\therefore (x-6)\,(x-2)=0$ and $x=6$ or 2. Do both values check?

13. $\dfrac{x-3}{x+1}+\dfrac{x+4}{x-2}=\dfrac{8x+2}{x^2-x-2}+1.$

14. $\dfrac{5}{(x-1)\,(x+2)}-\dfrac{2}{x^2-x-2}=\dfrac{8}{x^2-1}-\dfrac{5}{x^2-4}.$

15. $\dfrac{3-2x}{1-2x}-\dfrac{2x-5}{2x-7}=1-\dfrac{4x^2-2}{7-16x+4x^2}.$

16. $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}=\dfrac{6}{x^4-1}.$

17. $\dfrac{5x-8}{6x-15}-\dfrac{2x-5}{10x-4}=\dfrac{19x^2-29}{(2x-5)\,(15x-6)}.$

18. $\dfrac{1}{x-8}-\dfrac{1}{x-7}+\dfrac{1}{x-4}=\dfrac{1}{x-5},$

Hint. $\dfrac{1}{x-8}-\dfrac{1}{x-7}=\dfrac{1}{x-5}-\dfrac{1}{x-4},$

$\dfrac{x-7-x+8}{(x-8)(x-7)}=\dfrac{x-4-x+5}{(x-5)(x-4)},\qquad \dfrac{1}{(x-8)(x-7)}=\dfrac{1}{(x-5)(x-4)}.$

19. $\dfrac{7}{x-9}+\dfrac{2}{x-4}=\dfrac{7}{x-7}+\dfrac{2}{x-11}.$

Hint. $\dfrac{7}{x-9}-\dfrac{7}{x-7}=\dfrac{2}{x-11}-\dfrac{2}{x-4}.$

20. $\dfrac{1}{x-13}-\dfrac{2}{x-15}+\dfrac{2}{x-18}=\dfrac{1}{x-19}.$

Solve the following for x:

21. $a+\dfrac{b}{x}=c.$ Ans. $x=\dfrac{b}{c-a}.$

22. $\dfrac{a}{x}+b=\dfrac{b}{x}+a$

23. $\dfrac{x+a}{x-a}=\dfrac{5}{4}.$

24. $\dfrac{a-bx}{ax-b}=\dfrac{3}{4}.$

25. $\dfrac{m+x}{m+n}=\dfrac{m-x}{n-m}.$

26. $\dfrac{x+m}{3}-\dfrac{3}{x+m}=\dfrac{x-m}{3}.$

27. $\dfrac{x+ab}{x-ab}=\dfrac{a^2+ab+b^2}{a^2-ab+b^2}.$

28. $\dfrac{ax}{a+b}+\dfrac{bx}{a-b}=x+\dfrac{2b}{a}.$

29. $\dfrac{2x^2+ax+b}{3x^2+bx+a}=\dfrac{2}{3}.$

Read

CHAPTER XIII

THE FUNCTION, EQUATIONS, AND GRAPHING

157. In an algebraic expression involving several variables it is usually possible to find the value of one of these in terms of the others. It will be recalled that most formulas are written in such form as $i=prt$, $C=2\pi r$, $A=\pi r^2$, . . .

If such an equation as $3x+4y=7$ is solved for x, we have $x=\frac{1}{3}(7-4y)$. If it is solved for y, we have $y=\frac{1}{4}(7-3x)$.

If in an algebraic expression two variables are so related that when a value is given one then a value of the other is determined, the second variable is called a **function** of the first. In $C=2\pi r$, C is a function of r. Its value is fixed when a value is given r. Similarly, A is a function of r in $A=\pi r^2$, and x is a function of y in $x=\frac{1}{3}(7-4y.)$

An algebraic expression involving a single literal number is called a function of that literal number, for the value of the expression is determined by the value given that number. The expression x^3+x^2-x-4 is a function of x because the expression has a definite value for each value given x. The expression y^4-2y^2+4 is a function of y.

When it is necessary to refer to such an expression as x^3+x^2-x-4 several times during a discussion it is customary to represent the function in x by the symbol $f(x)$ which we read as the "f function of x."

If $f(x)$ is the expression x^3+x^2-x-4, then $f(a)$ in the same discussion will be the expression found by replacing x by a, or a^3+a^2-a-4. That is if $f(x)=x^3+x^2-x-4$, then $f(a)=a^3+a^2-a-4$ and $f(2)=2^3+2^2-2-4=6$.

Exercise 154

1. If $f(x)=x^2-x+1$, what is $f(y)$?

2. If $f(z)=z^3-3z^2+3z-1$, what is $f(k)$?

3. If $f(y)=y^3-27$, find $f(2)$, $f(0)$, and $f(-3)$.

4. If $f(x)=\frac{x-1}{x^2-x+1}$, find $f(1)$, $f(0)$, and $f(-1)$.

5. If $f(x)=x^6-x^4+x^2-1$, find $f(\frac{1}{2})$, and $f(-\frac{1}{2})$.

6. If $f(n)=n^3-3n^2-3n+11$, find $f(0)$, $f(-1)$, $f(-2)$, and $f(-\frac{1}{2})$.

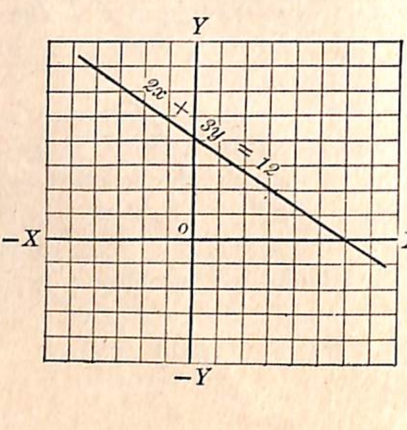
7. If $f(a)=a^3+3a^2+3a+2$, what is $f(-2)$? Therefore what binomial is a factor of $f(a)$ by the Factor Theorem?

8. If $f(x)=x^3-6x^2+11x-6$, show that $f(3)=f(2)=f(1)=0$, and name the three prime factors of $f(x)$.

9. If $f(x)=x^2-2x+1$, find $f(a-1)$, $f(a+1)$, and $f(a+y)$.

158. An equation of the first degree in two unknowns. It will be recalled (§ 63) that the **degree** of a **term** is determined by the number of literal prime factors that it contains provided that none of these appear in a denominator. The term $2a^2b$ is of the third degree. It is of the first degree in b and the second degree in a.

The **degree** of a **polynomial** with respect to, or in a given letter, is determined by its term of highest degree. The polynomial $a^3+3a^2b+3ab^2+b^3$ is of the third degree in a as well as in b.



It will be recalled that the graph of an equation of the first degree in two variables is a straight line. This explains the common name, "**linear equation.**" Note the graph of the equation $2x+3y=12$ in the accompanying figure.

If we express x as a function of y, we have $x=\frac{12-3y}{2}$. For every possible value of y there is a value for x. Similarly we may express y as a function of x, and show that for every possible value of x there is a value of y. The graph is the locus of a point under the condition imposed by the relation of x to y or of y to x. (See § 84.)

The equation $2x+3y=12$ is a true equality only under the condition imposed by this relation. It will be proved in analytic geometry that the locus of every equation of the first degree in two unknowns is a straight line.

159. An equation of the first degree in three or more unknowns.

As with the linear equation, so also with an equation of the first degree in three or more unknowns, there is a limitless number of sets of values for the unknowns that satisfy the equation.

Take the equation $3x+2y-z=5$ and note that each of the following sets of values satisfies it:

when $x=1$,	1,	1,	2,	2,	2,	2,	3,	3,
and $y=1$,	2,	3,	1,	-2,	0,	2,	3,	4,
then $z=0$,	2,	4,	3,	-3,	1,	5,	10,	12.

An equation of the first degree in two or more unknowns has a limitless number of solutions, and is therefore said to be **indeterminate.**

160. Two or more equations involving the same two or more unknowns are said to form a **system of equations.** We shall first discuss

Linear systems. If two equations of the first degree in the same two unknowns have no common pair of values they are said to be **incompatible,** or **contradictory,** and, if graphed, their straight lines will be found to be parallel such

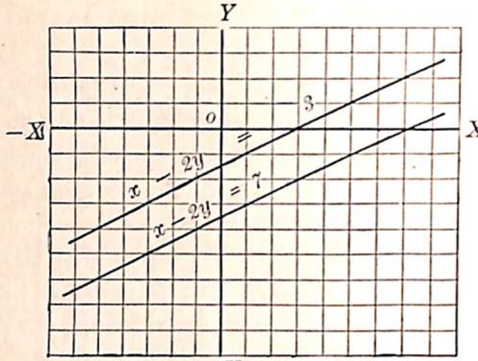


Fig. 1

as $x - 2y = 3$ and $x - 2y = 7$. (Fig. 1.)

Some systems have many pairs of values in common. When these are graphed their lines will be found to coincide, as $x - 2y = 5$ and $3x - 6y = 15$ in Fig. 2.

The second equation may be obtained from the first by multiplying both members by 3 and is called a **dependent equation.**

Two equations are said to be **independent** when one cannot be obtained from the other by any process that does not destroy the relation of the unknowns.

Two **independent linear equations** that have a single set of values in common, are said to form a **simultaneous linear system.** Their graphs will intersect in one point since two straight lines on a plane that are not parallel intersect in but one point. (See Fig. 3 where $2x+3y=6$ and $3x-y=5$ form such a system.)

161. The solution of a simultaneous linear system is completed when the common set of values for the unknowns is found. This may be accomplished by one of the following methods.

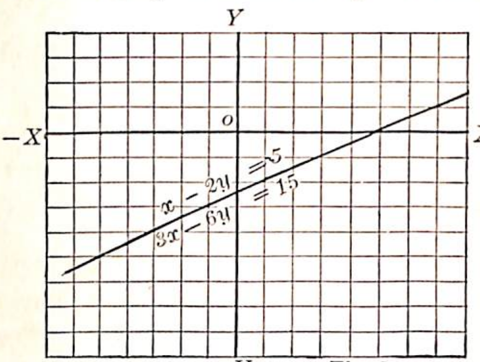


Fig. 2

Method I. Elimination of one of the unknowns by addition or subtraction. (See § 87.)

Rule. *Multiply the members of both equations by such numbers as will give equal absolute values for the coefficients of one of the unknowns. Add or subtract the resulting equations to eliminate that unknown and solve the resulting equation for the other unknown. Complete the solution.*

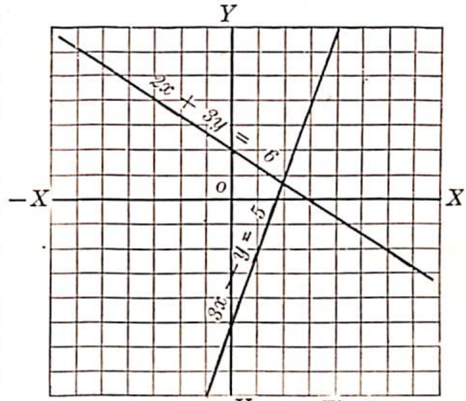


Fig. 3

Method II. Elimination of one of the unknowns by substitution. (See § 88.)

Rule. *Solve one of the equations for one unknown in terms of the other. Substitute the result for that unknown in the other equation and complete the solution.*

Method III. Elimination by comparison.

Rule. *In each equation find the value of the same unknown in terms of the other. Equate these two values and solve the resulting equation.*

Illustrative example. Given $2x+4y=12$ (1)
and $3x-2y=10$ (2)

From (1) $x=6-2y$ and from (2) $x=\frac{10+2y}{3}$

therefore $6-2y=\frac{10+2y}{2}$. Solving, gives $y=1$ and $x=4$.

Exercise 155

Solve the following systems by Method I and check:

1. $5y-2x=6$
   $8y-5x=-3.$

2. $5x+2y=19$
   $2x-5y=-4.$

3. $2a+3b=1$
$3a-2b=21.$

4. $5m+3n=6$
$3m-5n=7.$

5. $5x+11y=13$
$3x-7y=1.$

6. $x=\frac{1}{3}(9-y)$
$y=\frac{1}{2}(x+4).$

7. $x+1=\frac{1}{2}(x+y)$
$y-1=2(x-y).$

8. $9x=11+3y$
$6y=x-5.$

Solve the following systems by Method II and check:

9. $3R-4r=20$
$7R+11r=6.$

10. $5a-2b=3$
$4b-4a=-3.$

11. $7m-2n=\frac{19}{16}$
$5m+3n=\frac{1}{4}.$

12. $\frac{1}{2}x-\frac{1}{3}y=6$
$\frac{1}{4}x+\frac{1}{6}y=0.$

In solving such a system as is given in No. 12, any one of the three methods given for the solution of a simultaneous system in two unknowns may be employed, but the first method will be found most satisfactory.

Solve the following by Method III and check:

13. $\frac{1}{2}y-3x=2$
$y=14x.$

14. $5y=2x+1$
$8y=5x-11.$

15. $\frac{x}{2}+\frac{y}{3}-7=0$
$\frac{x}{3}+\frac{y}{2}-8=0.$

16. $3x-5y=4$
$5y-2x=\frac{-3}{2}.$

Graph both lines in each of the following sets, and locate carefully their point of intersection. Verify the accuracy of your graphs by finding the solution (if there is one) by Method I:

17. $3x-5y=1$
$4y+2x=8.$

18. $2x+3y=8$
$3x+\frac{9y}{2}=12$

19. $\frac{x}{2}+2y=3$
$3x+12y=4.$

20. $2x+5y=-12$
$3x-2y=1.$

21. $5x-y=4$
$2x+4y=-5.$

22. $3x-2y=-1$
$6x+8y=16.$

Solve the following systems by any method and check:

23. $x-\frac{7y}{5}=-23$
$x+\frac{y}{5}=17.$

24. $\frac{x+3}{2}+5y=9$
$\frac{y+9}{10}-\frac{x-2}{3}=0.$

25. $0.3x+0.2y=9.5$
$0.2x+0.3y=10.5.$

26. $.02x-.03y=60$
$.03x+.02y=155.$

Solve the following for x and y and check:

27. $x+y=2a$
$(a-b)x=(a+b)y.$

28. $cx-by=0$
$bx+cy=b^2+c^2.$

29. $x+my+m^2=0$
$x+ny+n^2=0.$

30. $x+ay=-1$
$y+c(x+1)=0$

31. $ax-by=a^2+b^2$
$(a-b)x+(a+b)y=2(a^2-b^2).$

While the following are not linear systems, they are simultaneous and can be solved conveniently by any of the three methods.

Do not clear of fractions until one unknown is eliminated. Solve each and check:

32. $\frac{10}{x}-\frac{9}{y}=8$
$\frac{8}{x}+\frac{15}{y}=-1.$

33. $\frac{3}{2x}-\frac{1}{2y}=7$
$\frac{4}{3x}-\frac{3}{2y}=2.$

34. $\frac{1}{x}-\frac{1}{y}=m$
$\frac{1}{x}+\frac{1}{y}=n.$

35. $\frac{a}{bx}+\frac{b}{ay}=a+b$
$\frac{b}{x}+\frac{a}{y}=a^2+b^2.$

162. Solution of a simultaneous linear system by determinants.

Let $a_1x+b_1y=c_1$ (1)
and $a_2x+b_2y=c_2$ (2) be a simultaneous linear system.

Multiplying both members of (1) by b_2 and both members of (2) by $-b_1$, we will have

$$a_1b_2x+b_1b_2y=b_2c_1 \quad (3)$$
$$-a_2b_1x-b_1b_2y=-b_1c_2 \quad (4).$$

Adding (3) and (4) and collecting gives
$(a_1b_2-a_2b_1)x=b_2c_1-b_1c_2$
and $x=\dfrac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$, if a_1b_2 is not equal to $(\neq)a_2b_1$.

Similarly, multiplying both members of (1) by $-a_2$ and of (2) by a_1 and adding gives

$(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$ and $y=\dfrac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$, if $a_1b_2\neq a_2b_1$.

The denominators of the fractional values of x and y are the same and may be written compactly in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, which is called a **determinant.** It is said to be of the second order since it has two rows and two columns. The letters a_1, a_2, b_1, and b_2 are called the **elements** of the **determinant** and a_1b_2, its **principal diagonal.** The value of a determinant of the second order is found by subtracting from the product of the elements that form its principal diagonal, the product of the other two elements.

Similarly, the numerators of the fractional values of x and y may be written as determinants, that for x being $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and for y being $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

The values of x and y may be written in determinant symbols and are $x=\dfrac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$, and $y=\dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

The numerator of the value of x is obtained from the denominator by substituting for the column $\begin{vmatrix} a_1 \\ a_2 \end{vmatrix}$, which are the coefficients of x in the given equations (1) and (2), the column of the known terms $\begin{vmatrix} c_1 \\ c_2 \end{vmatrix}$. Similarly, the numerator of the value of y is obtained from the denominator by replacing the column $\begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$ by $\begin{vmatrix} c_1 \\ c_2 \end{vmatrix}$.

The following **illustrative examples** may serve to show the possibilities arising from the use of determinants:

1. Solve $2x+4y=14$
$3x+y=11.$

$$x=\frac{\begin{vmatrix} 14 & 4 \\ 11 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}}=\frac{14-44}{2-12}=\frac{-30}{-10}=3.$$

$$y=\frac{\begin{vmatrix} 2 & 14 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}}=\frac{22-42}{-10}=\frac{-20}{-10}=2.$$

2. $x-3y=6$
$4x-5y=24.$

$$x=\frac{\begin{vmatrix} 6 & -3 \\ 24 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 4 & -5 \end{vmatrix}}=\frac{-30+72}{-5+12}=\frac{42}{7}=6.$$

$$y=\frac{\begin{vmatrix} 1 & 6 \\ 4 & 24 \end{vmatrix}}{7}=\frac{24-24}{7}=\frac{0}{7}=0.$$

Exercise 156

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & -3 \\ 2 & 4 \end{vmatrix}$. Ans. 18.

2. $\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$.

3. $\begin{vmatrix} 6 & -2 \\ 4 & -3 \end{vmatrix}$.

4. $\begin{vmatrix} -5 & -3 \\ -6 & -2 \end{vmatrix}$.

5. $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

6. $\begin{vmatrix} 2a & -4b \\ 3b & 7a \end{vmatrix}$.

7. $\begin{vmatrix} a+b & -b \\ b & a-b \end{vmatrix}$.

8. $\begin{vmatrix} 3a^2 & 2ab \\ 7ab & 4b^2 \end{vmatrix}$.

9. $\begin{vmatrix} 1 & x+y \\ x-y & y^2 \end{vmatrix}$.

10. $\begin{vmatrix} 2a+3b & a-b \\ 3a-2b & 2a-b \end{vmatrix}$.

Solve each of the following systems by determinants:

11. $2x-3y=-10$
$3x+5y=4.$

12. $5x+11y=13$
$3x-7y=1.$

13. $2x+5y=-14$
$3x-7y=8.$

14. $3x-5y=7$
$5x+3y=6.$

15. $5x+9y=28$
$7x+3y=20.$

16. $8x+9y=26$
$32x-3y=26.$

17. $21x-23y=2$
$7x-19y=12.$

18. $\frac{3}{2}x-7y=-38$
$\frac{3}{2}y-7x=-72.$

19. $\frac{1}{5}x-\frac{1}{3}y=-1$
$\frac{1}{5}y-\frac{1}{4}x=-2.$

20. $5x+7y=49$
$7x+5y=47.$

163. Systems of equations in more than two unknowns. We have already found that such an equation as $3x+2y-z=5$, (1), has a limitless number of sets of values that satisfy it. (See § 159.) It will be shown in analytic geometry that the locus of such an equation, with reference to three axes, x, y, and z, each perpendicular to the other two, is a plane in space. Such an equation as $3x+2y-z=8$, (2), has no set of values in common with (1) and they are said to be **incompatible.** The locus of (2) is a plane parallel to that of (1). If, however, we take such an equation as $2x-3y+z=7$, (3), (1) and (3) will be found to have many sets of values in common, for adding (1) and (3) gives $5x-y=12$, which is a linear equation and has a limitless number of sets of values.

Equations (1) and (3) are said to be **independent equations** since neither can be obtained from the other by any process that does not destroy the relation among the unknowns.

Three independent equations in the same three unknowns that form a **simultaneous system** have but a single set of values that are common for all three. We must have as many independent equations as there are unknowns if we are to find the solution of the system.

The system $3x+2y-z=5$ (1)
$2x-3y+z=7$ (3)
and $x+y+z=-3$ (4) may be solved by any one of the three methods given in § 161 but Method I is

very convenient, for adding (1) and (3) gives $5x-y=12$ (5) and adding (1) and (4) gives $4x+3y=2$ (6).

Equations (5) and (6) form the simultaneous linear system
$5x-\ \ y=12$ (5)
$4x+3y=\ 2$ (6) which may be solved by Method I by multiplying (5) by 3, and adding the product to (6) gives $19x=38$, or $x=2$. Therefore $y=-2$ and $z=-3$.

Exercise 157

Solve the following simultaneous systems:

1. $3x-2y+z=8$
$2x-3y-2z=-1$
$x+y+z=3.$

2. $3x-4y+6z=60$
$4x-5y+4z=20$
$10x-12y+15z=180.$

3. $3x+5y+6z=31$
$3x+4y+5z=26$
$2x+3y+4z=20.$

4. $x+y=5$
$y+z=7$
$x+z=9.$

5. $a+2b-3c=6$
$2a+4b-7c=9$
$3a-b=23.$

6. $7x+y=4z+w$
$x+w=y$
$2z+3y=15+w$
$3y+8=7x+2z+3w.$

7. $2x-y=8$
$3y-z=13$
$4z-w=16$
$5w-x=13.$

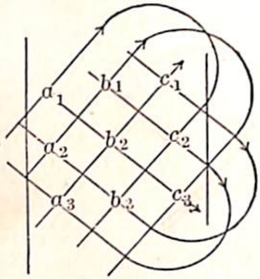
8. $x+y+z=6$
$y+z+w=9$
$x+z+w=8$
$x+y+z+w=10.$

9. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=24$
$\frac{1}{x}+\frac{1}{y}=14$
$\frac{2}{x}-\frac{1}{z}=2.$

10. $\frac{1}{x}-\frac{2}{y}+3=0$
$\frac{1}{y}-\frac{3}{z}+4=0$
$\frac{1}{z}-\frac{4}{x}+2=0.$

164. Simultaneous systems involving three or more unknowns may be solved very conveniently by determinants obtained in the same manner as were those of § 162. A system with three unknowns will involve a determinant of the third order with three rows and three columns; one with four unknowns, a determinant of the fourth order. Determinants above the third order are best studied in a more advanced course.

A **determinant** of the **third order** may be defined as a compact method for indicating six products of three factors each, three of which are positive and three negative.

Determinant 

$\equiv a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$
$- a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$

The arrows indicate how the products are obtained and their character.

Illustrative example.

1. Solve
$$\begin{aligned} 2x+3y+\ z&=11\\ x+2y+3z&=14\\ 3x+\ y+2z&=11. \end{aligned}$$

$$x=\frac{\begin{vmatrix} 11 & 3 & 1\\ 14 & 2 & 3\\ 11 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} 2 & 11 & 1\\ 1 & 14 & 3\\ 3 & 11 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} 2 & 3 & 11\\ 1 & 2 & 14\\ 3 & 1 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2 \end{vmatrix}}.$$

Note that the determinant denominators are the same for all three unknowns and are the columns of the coefficients of x, y, and z in their order. Note that the numerators are obtained from the denominators as in § 162 by replacing the column of the coefficients of x, y, or z by the column of the constants $\begin{vmatrix} 11\\ 14\\ 11. \end{vmatrix}$

The value of the common determinant denominator is obtained by following the terms of the definition of a determinant of the third order.

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 2\cdot2\cdot2+1\cdot1\cdot1+3\cdot3\cdot3-3\cdot2\cdot1$$

$$-1\cdot3\cdot2-2\cdot1\cdot3=8+1+27-6-6-6=18.$$

The determinant numerator for $x=11\cdot2\cdot2+14\cdot1\cdot1+11\cdot3\cdot3$

$$-11\cdot2\cdot1-14\cdot3\cdot2-11\cdot1\cdot3=44+14+99-22-84-33=18$$

$\therefore x=\frac{18}{18}=1$. Similarly, $y=\frac{36}{18}=2$ and $z=\frac{54}{18}=3$.

Exercise 158

Evaluate the following determinants:

1. $\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 3 \\ 1 & 4 & 2 \end{vmatrix}$. Ans. 31.

2. $\begin{vmatrix} 5 & 1 & 6 \\ 3 & 4 & 2 \\ 2 & 3 & 3 \end{vmatrix}$.

3. $\begin{vmatrix} 2 & 4 & -2 \\ 0 & 3 & 3 \\ -1 & -5 & 0 \end{vmatrix}$.

4. $\begin{vmatrix} 5 & 10 & -3 \\ 3 & 6 & 2 \\ 2 & 4 & -8 \end{vmatrix}$.

5. $\begin{vmatrix} a & 3 & -3 \\ b & 2 & 4 \\ 2a & -1 & 2 \end{vmatrix}$.

6. $\begin{vmatrix} a & b & b \\ b & a & a \\ a & b & a \end{vmatrix}$.

Solve the following simultaneous equations by determinants:

7. $2x+3y-2z=2$
$3x-y+z=4$
$x+2y-3z=-4.$

8. $x+y+z=6$
$2x-y-z=3$
$3x+4y+3z=16.$

9. $2a-b+c=12$
$3a+2b-2c=-10$
$5a+3b+c=6.$

10. $x+y+z=-5$
$2x+3y-3z=5$
$3x-2y+4z=-18.$

11. $2x+3y+4z=19$
$5x-2y-3z=4$
$7y+5z=0.$

12. $4x+3y=3$
$6y-2z=7$
$5x+3z=-5.$

165. Solution of problems.

Exercise 159

Some of the problems of algebra may be classified according to the essential feature on which it is best to build the necessary equations.

I. **Work problems.** Essential feature: what fraction of the whole work is completed in some unit of time?

1. If A works alone, he can lay a cement walk in 10 days. If B works with him, they can lay the walk in 6 days. How many days would it take B working alone?

Let x represent the number of days that B would require if working alone.

In one day A can do $\frac{1}{10}$ of the work and B, $\frac{1}{x}$ of the work and both together will complete $\frac{1}{10}+\frac{1}{x}$ of the whole.

But according to the problem they will finish $\frac{1}{6}$ of the walk in one day, therefore the equation, $\frac{1}{10}+\frac{1}{x}=\frac{1}{6}$.

2. Two men if they work together will finish a certain task in 10 hours. If, at the end of the 6th hour, one man is withdrawn the other will finish 12 hours later. How many hours would each require if working alone?

The equations are $\frac{1}{x}+\frac{1}{y}=\frac{1}{10}$ (1)

and $\frac{6}{x}+\frac{6}{y}+\frac{12}{y}=1$ (2).

Can you explain (1) as in No. 1 and (2) by the axiom that the whole is equal to the sum of all of its parts?

3. Water may enter a tank through three pipes A, B, and C. If the tank is empty and all the pipes are opened, the tank will be filled in $2\frac{2}{3}$ hours. If A and B alone are opened, it will be filled in $3\frac{3}{7}$ hours. If B and C alone are opened, it will be filled in $4\frac{4}{5}$ hours. How many hours would each pipe require if opened alone?

The first equation is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2\frac{2}{3}}=\frac{3}{8}$.

4. A tank has one outlet pipe and two inlets. If the tank is empty and all are opened, it will be filled in 12 hours. If the outlet is closed, the two inlets will fill it in 6 hours. If the smaller inlet is closed and the outlet opened the larger inlet will fill the tank in 60 hours. If the tank is full and the inlets closed how long will it take the outlet to empty it? If empty and the outlet closed, how long would it take each inlet alone to fill it? What would happen if the tank were full and the outlet and the smaller inlet were opened?

II. **Mixture problems.** Essential feature: what part, or per cent, some element is of the whole mixture.

5. How much salt must be added to a 2% salt solution weighing 20 lbs. to make it a 5% solution?

The equation is $\frac{2}{100}(20)+x=\frac{5}{100}(20+x)$.

Can you explain it?

6. A druggist has an acid in two strengths, one 70% pure and the other 95% pure. How much of each must he take to make 20 liters 85% pure?

The equations are $x+y=20$ and $\frac{70}{100}x+\frac{95}{100}y=\frac{85}{100}\,20$.

7. How many quarts of milk testing 4% butter fat must be added to 20 quarts of cream testing 22% butter fat to make the cream test 16%?

The equation is $\frac{4}{100}x+\frac{22}{100}20=\frac{16}{100}(x+20)$.

III. **Digit problems.** Essential feature: if letters are used for digits the place of the digit must be accounted for by multiplying by 10, 100 . . . . .

8. The sum of the digits of a two-digit number is 12 and, if the digits were interchanged, the resulting number would be 18 more than the given number. What is the number?

The equations are $x+y=12$ and $10x+y+18=10y+x$. Explain these.

9. The sum of the digits of a three-digit number is 14 and the units' digit is equal to the sum of the other two. If the tens' and hundreds' digits were interchanged, the number would be increased by 270. What is the number?

The equation from the last condition is $100x+10y+z+270 = 100y+10x+z$. Can you explain?

IV. **Uniform motion problems.** Formula, $d=tr$.

10. A freight train running 30 miles per hour is 50 miles ahead of a passenger train running 40 miles per hour. In how many hours will the passenger train overtake the freight train? Solve by a graph before solving by an equation.

11. A train running 30 miles per hour requires 20 minutes longer to go a certain distance than a train running 40 miles per hour. Find the distance.

12. Two automobiles start together in the same direction around a circular track. One can make the circuit in $2\frac{1}{2}$ minutes and the other in 3 minutes. In how many minutes will they be together again?

Hint. The faster machine must gain one circuit and the equation is $\frac{x}{2\frac{1}{2}}-\frac{x}{3}=1.$

V. **Lever problems.** Essential feature, Law of Levers, $Lw=lW$, or the weight on one arm times its distance from the fulcrum equals the weight on the other arm times its distance from the fulcrum.

13. Assuming that the bar of the lever itself has no appreciable weight, how far from the fulcrum on one side must a weight of 100 lbs. be placed to balance a weight of 80 lbs. placed five feet from the fulcrum on the other side?

The equation is $5 \cdot 80 = 100x$. Explain by a figure.

14. A lever 16 feet long, with its fulcrum at the center, has a weight of 12 lbs. hung at one end and 8 lbs. hung at the other. Where must an additional weight of 8 lbs. be hung to provide an exact balance?

At the first reading of a problem the student should ascertain (1) if it belongs to some particular group, (2) how many unknowns are involved, (3) if there is sufficient data for the necessary equations.

15. How much cream testing 16% butter fat must be added to 40 quarts of milk testing 3% butter fat to make the milk pass the legal test of 4% butter fat?

16. A works three-fourths as fast as B and both together can finish a task in $5\frac{1}{7}$ hours. How long would it take each if he works alone?

17. A certain government has two kinds of old coins, one 95% silver and 5% copper, the other 80% copper and 20% silver. How much of each must be used to make 2000 lbs. of metal for coinage 92% silver?

18. The sum of the digits of a two-digit number is 9 and if the digits were interchanged, the number would be increased by 27. What is the number?

19. A lever 12 feet long, having a weight at each end, is balanced at a point $4\frac{1}{2}$ feet from one end. If the weight on the shorter arm is 100 lbs., what is the weight on the longer arm?

20. A man invests $12,000 in two kinds of bonds, a part in 6% bonds at 92 and the rest in 5% bonds at 70. If his annual interest is $800, what is his investment in each kind?

Hint. If x is the number of dollars that he invests in the first kind, then $x/92$ is the number of those bonds that he buys and $6x/92$ is his annual interest from them.

21. A merchant has an acid in two strengths such that 10 quarts of the first kind and 8 of the second makes a mixture 80% pure, while 7 quarts of the first and 2 of the second makes a mixture 84% pure. What is the % of purity of each kind?

22. A man invests a certain sum in 5% bonds at 90 and twice as much in 4% bonds at 80. If his annual interest from the investment is $560, what sum did he put into each kind?

23. How much pure copper must be added to 500 lbs. known to be 90% copper to make a metal 96% pure?

24. How much tin must be added to 2000 lbs. of metal testing 92% copper and 8% tin to make the mass 88% copper?

25. A and B can do a task in $5\frac{5}{11}$ hours, that A and C could do in $6\frac{2}{3}$ hours, and B and C in $7\frac{1}{2}$ hours. How many hours would it take each if he worked alone?

26. A camp equipment of 90 lbs. is swung on a pole between two boys. The boys are 10 feet apart and the weight is suspended 4 feet from one boy. What weight does each carry?

27. The sum of the digits of a three-digit number is 10. If the tens' and units' digits are interchanged the number is increased by 27. If the units and hundreds digits are interchanged it is increased by 198. What is the number?

28. A man invests a certain part of a sum of money in 5% bonds at 90 and the rest in 4% at 80 and his annual interest is $992. If the first part had been invested in the 4% bonds at 80 and the rest in the 5% bonds at 90, his interest would have been $984. What amount does he invest in each?

29. A messenger starts from a camp at the rate of 10 miles per hour, and 15 minutes later a second messenger starts after the first at the rate of 15 miles per hour. In what time will the second overtake the first?

30. If a number of two digits is divided by the sum of its digits the quotient is 7 and the remainder 3. What is the number if the digit in units' place is 3 less than the digit in tens' place?

31. A certain number is added to each of the numbers 2, 9, 3, and 7. If the product of the first two results equals the product of the second two, what is the number that is added?

32. A country grocer sells Mrs. Brown 3 pounds of butter and 4 dozen eggs for $3.60. He buys from Mrs. Jones 4 pounds of butter and 5 dozen eggs for $4.00. Find the grocer's retail price on each if he makes a profit of 10 cents a pound on butter and 5 cents a dozen on eggs.

33. A grocer wishes to mix 60 lbs. of coffee at a total cost of $16.00. If he uses two grades of coffee, one costing 36 cents and the other 20 cents a pound, how many pounds of each must he use?

34. The sum of the digits of a two-digit number is 12. The quotient of the number divided by the units' digit is 6. What is the number?

35. A chemist has an acid in two strengths, one 85% pure and the other 60% pure. He has an order for 15 ounces of the acid 75% pure. How many ounces of each kind must he use to fill the order?

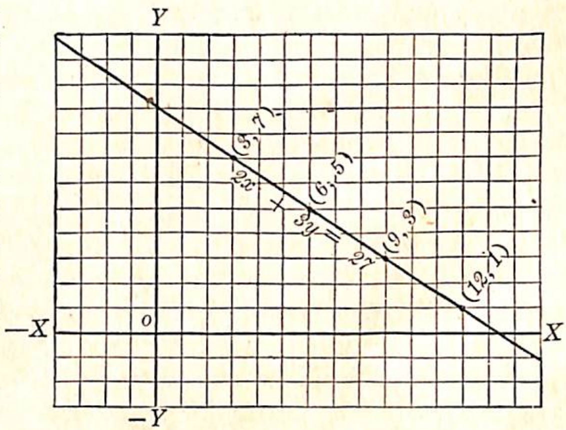
166. Applications of the indeterminate equation.

Exercise 160

1. Find all positive integral sets of values for $2x+3y=27$.

Expressing x as a function of y, $x=\frac{27-3y}{2}=13-y+\frac{1-y}{2}$.

Now if the value of x is to be an integer such values of y must be used as will make $\frac{1-y}{2}$ zero or an integer. If $y=1$, the fraction becomes zero. If $y=3$, it becomes -1. Therefore when $y=1, 3, 5, 7,$ then $x=12, 9, 6, 3$.



There are four sets of positive integral values. A carefully constructed graph will pass through the four points as indicated in the figure.

2. In how many ways can a debt of $89 be paid with $2 and $5 bills?

3. A man wishes to pay a bill of $2.23 in cents, dimes, and quarters using the same number of cents as of dimes. In how many ways can he do this?

4. A child's bank is found to contain $2.84 in cents, nickels, and dimes. If there are 60 coins in the bank, in how many ways can the amount be made up?

5. A farmer makes a shipment of chickens, ducks, and turkeys, 45 fowls in all. The shipment nets him $120, the chickens netting $1.00 each, the ducks $2.50, and the turkeys $4.50. In how many ways may the shipment have been made up?

6. A wholesale grocer has coffee in one pound cans which he sells at 25 and 35 cents, respectively. He fills an order for $22.55 made up of cans from both varieties. In how many ways can he do this? How many of each kind would he have shipped if the purchaser had requested that the number of each be approximately the same?

CHAPTER XIV

EXPONENTS, RADICALS, AND IMAGINARIES

167. Exponents. Define each of the following: power, root, square root, cube root, term, like or similar terms.

We have defined an **exponent** as a number so placed as to indicate how many times another number is to be taken as a factor. All exponents so far in our study of algebra have been positive integers or, when literal numbers, have been assumed to be positive integers. It is the purpose of this chapter to expand our knowledge of exponents to include the whole field of real numbers, i.e., zero, negative numbers, and fractions, as well as positive integers.

We have learned and used the laws of exponents (see pages 68, 197, and 200), which were conveniently stated in the following **typeforms** or **formulas**:

I. For multiplication, $a^m \cdot a^n \equiv a^{m+n}$ and $(a^m \cdot b^n)\ (a^x \cdot b^y) \equiv a^{m+x}b^{n+y}$.

II. For division, $a^m \div a^n \equiv a^{m-n}$ and $(a^m b^n) \div (a^x b^y) \equiv a^{m-x} b^{n-y}$.

III. For finding a power, $(a^m)^n \equiv a^{mn}$, and $(a^m b^n)^x \equiv a^{mx} b^{nx}$.

The typeform for finding a root has not been given previously, but the rule may be found on page 200 and the typeform will be made evident by the discussion and use of fractional exponents in this chapter.

IV. For finding a root, $\sqrt[n]{a^m} \equiv a^{\frac{m}{n}}$, and $\sqrt[x]{a^m b^n} \equiv a^{\frac{m}{x}} b^{\frac{n}{x}}$.

For convenience we repeat the proof of the rule for finding the exponent in multiplication. It is as follows:

$a^m \cdot a^n \equiv (a \cdot a \cdot a \cdot a\ .\ .\ .$ to m factors$) \cdot (a \cdot a \cdot a \cdot a\ .\ .\ .$ to n factors$) \equiv a \cdot a \cdot a \cdot a \cdot a \cdot a\ .\ .\ .$ to $m+n$ factors, or a^{m+n}.

State the rule for finding the exponent of a letter in the product of several terms involving different literal factors.

Make the proof for it similar to that just given for $a^m \cdot a^n$. Similarly, prove both typeforms for division and state the rules.

Prove the typeforms for finding a power and state the rules.

Exponents are very convenient for writing compactly very large numbers. For instance it is said that the cost of the Great War to the world was \$200,000,000,000. This may be written $\$2\cdot 10^{11}$.

The "light year" of the astronomer is the distance that a ray of light will travel in one year, or $365\cdot 24\cdot 60\cdot 60\cdot 186000$ miles. (A ray of light travels 186000 miles in a second of time.) This distance is a little less than $58657\cdot 10^8$ miles. Check by actual multiplication.

Exercise 161

Write at sight the result for each of the following:

1. $(a^5)^3$. **2.** $(a^m)^2$. **3.** $(b^5)^x$. **4.** 10^7.

5. $(m^a)^b$. **6.** $(a^{mx})^2$. **7.** $(a^{mx})^n$. **8.** $(a^m)^m$.

9. $(x^{2a})^{3a}$. **10.** $(a^m)^m\cdot(a^n b^x)$. **11.** $(a^2b^n)\cdot(a^n b^2)$.

12. $(3^2\cdot 2^3)\cdot(3^3\cdot 2^2)$. **13.** $(2^2\cdot 3^3\cdot 5^5)^2$. **14.** $(a^x b^x)^n\cdot(a^n b^n)^x$.

15. $(a^5b^4)\div(a^4b^3)$. **16.** $(3ab^2c^3)\div(abc^2)$.

17. $(5^5\cdot 3^4\cdot 2^3)\div(5^2\cdot 3^2\cdot 2^3)$. **18.** $(24a^n b^n)\div(3a^2b^3)$.

19. $(a^4b^3c^2)\div(a^2b^2c)\cdot(abc^2)\div(a^2bc^2)$.

20. $(x^3y^2z)\div(xyz)\cdot(x^2yz^2)\div(xy^2z)\cdot(x^2yz)$.

21. $(x^a b^y)^2\cdot(x^a b^y)^3\div(x^2b^3)^a\cdot(x^3b^2)^y$.

168. The zero exponent and its meaning.

According to the typeform for division, $a^3\div a^3=a^0$. Similarly, $a^m\div a^n$ gives a^0 when $m=n$. If the division of a^3 by a^3 is performed in fractional form the meaning of a^0 is evident for $\frac{a^3}{a^3}=1$. $\therefore a^0=1$. Or, by the typeform for multiplication $(a+b)^m\cdot(a+b)^0=(a+b)^m$. But $(a+b)^m\cdot 1=(a+b)^m$. $\therefore (a+b)^0=1$. Hence we have the **principle:**

Any number expression whose exponent is 0 is numerically equal to 1.

Exercise 162

Simplify each of the following:

1. $a^2 \cdot a^0 \cdot a^3$. 2. $a^2 \cdot b^0 \cdot c^3 \cdot d^0$. 3. $(2a)^2 \cdot (2a)^0 \cdot (2a)^3$.

4. $(a^x b^y c^z d^w)^0$. 5. $(ax-by)^2 \cdot (ax-by)^0 \cdot (ax-by)^3$.

Find the numerical value for each of the following:

6. $5^2-4^0+3^2 \cdot 3^0-2^0 \cdot 2^3+2^2$.

7. $(4a-2)^0+7 \cdot 2^0-(7 \cdot 2)^0-(6-3a)^0+4(4^2-3^3)^0$.

8. $(\frac{1}{2})^2-(\frac{3}{2})^0+4 \div (\frac{1}{2})^2 \cdot 2^0 \div (2 \cdot 3^2 \cdot 4^3 \div 8^2)^0+(4\frac{3}{8}-2\frac{1}{2})^0$.

169. The negative exponent and its meaning.

From the typeform for division we have $a^2 \div a^3 = a^{-1}$ and, in general, $a^m \div a^n = a^{m-n}$ gives a negative exponent when m is less than n.

When the division is placed in the form of a fraction and reduced to its simplest form, the result is as follows:

$\frac{a^2}{a^3} = \frac{1}{a}$. (Dividing both members by a^2.) Evidently, since $a^2 \div a^3$ gives a^{-1} and $\frac{a^2}{a^3}$ gives $\frac{1}{a}$, then $a^{-1} \equiv \frac{1}{a}$. In general $a^x \div a^{x+y} = a^{-y}$ and $\frac{a^x}{a^{x+y}} \equiv \frac{a^x}{a^x \cdot a^y} = \frac{1}{a^y}$. $\therefore a^{-y} \equiv \frac{1}{a^y}$.

Hence the **principle:**

The minus sign before an exponent indicates that the reciprocal of the expression affected by the exponent is to be taken with the exponent positive.

For methods for clearing a term of negative exponents study the following:

Illustrative examples.

1. $2a^{-1} = 2 \cdot \frac{1}{a} = \frac{2}{a}$.

2. $a^2 b^{-3} c = a^2 \cdot \frac{1}{b^3} \cdot c = \frac{a^2 c}{b^3}$.

3. $\frac{2a^{-1}b}{x^{-2}y} = \frac{2 \cdot \frac{1}{a} \cdot b}{\frac{1}{x^2} y} = \frac{\frac{2b}{a}}{\frac{y}{x^2}} = \frac{2bx^2}{ay}$.

Example 3 makes evident the following rule:

A factor may be moved from numerator to denominator, or from denominator to numerator, without changing the value of the fraction if the sign of the exponent of the factor is changed.

Negative exponents furnish a convenient method for expressing small decimals. For instance $.00005=\frac{5}{100000}=5\cdot 10^{-5}$.

Exercise 163

Clear each of the following of negative exponents and simplify if possible:

1. $2a^{-3}bc^{-1}$. Ans. $\frac{2b}{a^3c}$.
2. $3^{-1}ab^{-1}$.
3. $a^{-1}bc^{-1}d$.
4. $\frac{2a^{-1}}{bc^{-1}}$.
5. $\frac{3^{-1}ab^{-1}c}{2a^{-1}b^2c^{-2}}$.
6. $\frac{m(a-b)^{-1}}{x}$.
7. $\frac{2(a-b)^{-1}}{3^{-1}(c-d)^{-2}}$. Ans. $\frac{6(c-d)^2}{a-b}$.
8. $\frac{3a^{-1}(x-y)^{-2}c}{2^{-2}a^{-3}(x-y)c^{-2}}$.
9. $\frac{a^{-1}-b^{-1}}{a^{-1}+b^{-1}}=\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}=\frac{b-a}{b+a}$.
10. $\frac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$.
11. $\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}$.
12. $\frac{x^{-2}-y^{-2}}{x^{-3}-y^{-3}}$.
13. $\frac{x^{-1}-2y^{-1}}{x^{-2}-4y^{-2}}=\frac{\frac{1}{x}-\frac{2}{y}}{\frac{1}{x^2}-\frac{4}{y^2}}=?$
14. $\frac{a^{-1}-3b^{-1}}{a^{-3}-27b^{-3}}$.
15. $\frac{4^{-1}-3^{-1}}{4^{-1}+3^{-1}}$. Ans. $-\frac{1}{7}$.
16. $\frac{3^{-1}-5^{-1}}{3^{-3}-5^{-3}}$.
17. $2^{-2}\cdot 3\cdot 4\cdot 6^{-2}$.
18. $\frac{2^a\cdot 2^{-3}-2^{-2}}{2^a\cdot 2^{-2}-2^{-1}}$.
19. $5^{-2}\cdot 25\cdot 2^{-3}\left(\frac{1}{4}\right)^{-2}\cdot 2$.

Write without denominators:

20. $\dfrac{2a}{bc}=2ab^{-1}c^{-1}$.
21. $\dfrac{3xy}{2m^2}$.
22. $\dfrac{3a}{a-b}=3a(a-b)^{-1}$.
23. $\dfrac{3xy}{x-y}$.
24. $\dfrac{2bc}{b(b-c)^2}$.
25. $\dfrac{3}{x^2-2xy+y^2}$.
26. $\dfrac{a^2-2ab+b^2}{m^2+mn+n^2}$.

Exercise 164

First perform the indicated operation, then clear the result of negative exponents.

1. $(x^{-1}-y^{-1})^2$.
2. $(x^{-2}-2x^{-1}y^{-1}+y^{-2})\,(x^{-1}-y^{-1})$.
3. $(a^{-1}-b^{-1})^3$.
4. $(x^{-2}-x^{-1}-6)\,(x^{-1}-2)$.
5. $(x^{-2}-4)\div(x^{-1}-2)$.
6. $(x^{-3}-y^{-3})\div(x^{-1}-y^{-1})$.
7. $(x^{-3}+y^{-3})\div(x^{-1}+y^{-1})$.
8. $(a^{-5}+b^{-5})\div(a^{-1}+b^{-1})$.
9. $(x^{-3}-3x^{-2}+3x^{-1}-1)\div(x^{-1}-1)$.
10. $(x^{-4}-x^{-2}+2x^{-1}-1)\div(x^{-2}-x^{-1}+1)$.
11. $(x^{-2}-2x^{-1}+1)^2$.
12. $(x^{-1}y^2-x^2y^{-1})^3$.
13. $(2a^{-1}-7-3a)\,(4a^{-1}+5)$.
14. $(a^{-1}b+2+ab^{-1})\,(a^{-1}b-2+ab^{-1})$.

170. Meaning of the fractional exponent.

Since $x^{\frac{1}{2}}\cdot x^{\frac{1}{2}}=x^{\frac{1}{2}+\frac{1}{2}}=x^{\frac{2}{2}}$ or x (Typeform I), therefore $x^{\frac{1}{2}}$ is one of the two equal factors of x, or $x^{\frac{1}{2}}$ is the square root of x, i.e. $x^{\frac{1}{2}}=\sqrt{x}$. Similarly $x^{\frac{1}{3}}\cdot x^{\frac{1}{3}}\cdot x^{\frac{1}{3}}=x^{\frac{3}{3}}$ or x. $\therefore x^{\frac{1}{3}}=\sqrt[3]{x}$. Also $x^{\frac{2}{3}}\cdot x^{\frac{2}{3}}\cdot x^{\frac{2}{3}}=x^{\frac{6}{3}}$ or x^2. $\therefore x^{\frac{2}{3}}=\sqrt[3]{x^2}$. But $x^{\frac{2}{3}}=(x^{\frac{1}{3}})^2$. $\therefore x^{\frac{2}{3}}=(\sqrt[3]{x})^2$ and $\sqrt[3]{x^2}=(\sqrt[3]{x})^2$.

Hence the **principle:**

The denominator of a fractional exponent indicates what root of the number is to be found and the numerator indicates what power.

Exercise 165

Write with fractional exponents and simplify where possible:

1. $\sqrt{2}$. 2. $\sqrt[3]{3}$. 3. $2\sqrt{a}$. 4. $3\sqrt[3]{a^2}$.

5. $2\sqrt{2}$. Ans. $2^{\frac{3}{2}}$. 6. $\sqrt[6]{a^5}$. 7. $3\sqrt[3]{(a+b)^5}$.

8. $\sqrt[n]{x^m}$. Ans. $x^{\frac{m}{n}}$. 9. $2a\sqrt[3]{x^2y}$. Ans. $2ax^{\frac{2}{3}}y^{\frac{1}{3}}$.

10. $3\sqrt[3]{x^3y}$. 11. $ab\sqrt{a^2b}$. Ans. $a^2b^{\frac{3}{2}}$.

12. $ab\sqrt[3]{a^2b}$. 13. $3(a+b)\sqrt{a+b}$.

14. $2xy\sqrt[3]{x^2y}$. 15. $2x\sqrt[3]{x^{-2}y}$.

16. $4ab\sqrt{2a^{-1}b^{-2}}$. Ans. $2^{\frac{5}{2}}a^{\frac{1}{2}}$.

17. $3(a+b)\sqrt{3(a+b)}$. 18. $(m-n)^2\sqrt[3]{(m-n)^{-2}}$.

Write as radicals:

19. $a^{\frac{1}{2}}=\sqrt{a}$. 20. $b^{\frac{2}{3}}$. 21. $c^{\frac{3}{4}}$. 22. $x^{\frac{m}{n}}$.

23. $2a^{\frac{2}{3}}$. Ans. $2\sqrt[3]{a^2}$. 24. $(3ab)^{\frac{2}{3}}$. 25. $(3ab)^{\frac{4}{3}}$.

26. $2xy^{\frac{1}{3}}z^{\frac{2}{3}}$. Ans. $2x\sqrt[3]{yz^2}$. 27. $x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{4}{3}}$. 28. $x^{\frac{a}{b}}y^{\frac{a}{b}}$.

Write as radicals and without negative exponents:

29. $a^{-\frac{2}{3}}=\frac{1}{\sqrt[3]{a^2}}$. 30. $a^{-\frac{1}{2}}b^{\frac{1}{2}}$. Ans. $\sqrt{\frac{b}{a}}$.

31. $2^{-\frac{1}{2}}a^{\frac{1}{2}}x^{-\frac{1}{2}}$. 32. $(a+b)^{-\frac{1}{2}}$. 33. $3(x+y)^{-\frac{2}{3}}$.

34. $2^{-\frac{2}{3}}a^{\frac{1}{3}}(b-c)^{-\frac{1}{3}}$. 35. $3a^{\frac{1}{2}}b^{-\frac{3}{5}}c^{\frac{2}{3}}$.

Write without denominators or radical signs:

36. $\frac{2}{\sqrt{x-y}}=2(x-y)^{-\frac{1}{2}}$. 37. $\frac{2a\sqrt{y}}{\sqrt{x}\sqrt[3]{y}}$. 38. $\frac{3ab}{\sqrt{a(a+b)}}$.

39. $\frac{2a}{3b(x-y)^3}$. 40. $\frac{abc}{\sqrt[3]{a^2+2ab+c^2}}$. 41. $\frac{xy}{x^{-1}+y^{-1}}$.

171. Principal root. Since $(+2)^2=4$ and $(-2)^2=4$, it is evident that 4 has two square roots, $+2$ and -2. Similarly every number has two square roots. We shall also discover that every number has three cube roots, four fourth roots, and five fifth roots. Some of these roots will be found to be neither rational nor irrational, but will require a new kind of number. (See § 180.)

When there are two real roots they are equal in absolute value, but opposite in character, as $+2$ and -2, the square roots of 4.

It is convenient to define the **principal root** of a number as its real root if there is but one real root and as its positive real root if there are two real roots, that is $\sqrt{4}$, or $4^{\frac{1}{2}}=2$, $\sqrt[3]{8}$ or $8^{\frac{1}{3}}=2$, $\sqrt[3]{-27}$ or $(-27)^{\frac{1}{3}}=-3$, and $\sqrt[5]{-32}$ or $(-32)^{\frac{1}{5}}=-2$.

Hereafter, unless otherwise stated, the radical sign and the fractional exponent will indicate that the principal root only is to be taken.

It is necessary to keep in mind that the root of a number is one of its equal factors, two if a square root, three if a cube root . . . , but the roots of an equation are the values of the unknowns that satisfy the equation, one if the equation is of the first degree in a single unknown, and two if it is of the second degree.

172. Evaluation of arithmetical expressions with fractional exponents.

Exercise 166

Find the value of each of the following:

1. $9^{\frac{1}{2}}$. **2.** $8^{\frac{2}{3}}$. Ans. 4. **3.** $4^{\frac{3}{2}}$.
4. $9^{\frac{3}{2}}$. **5.** $8^{\frac{5}{3}}$. **6.** $9^{-\frac{3}{2}}$. Ans. $\frac{1}{27}$.
7. $8^{-\frac{5}{3}}$. **8.** $9^{-\frac{5}{2}}$. **9.** $8^{-\frac{4}{3}}$. **10.** $(\frac{1}{4})^{-\frac{1}{2}}$.
11. $(\frac{1}{8})^{-\frac{2}{3}}$. **12.** $(\frac{1}{4})^{-\frac{5}{2}}$. **13.** $81^{\frac{1}{4}}$. **14.** $125^{-\frac{2}{3}}$.
15. $(\frac{1}{64})^{-\frac{2}{3}}$. **16.** $(64)^{\frac{5}{6}}$. **17.** $(.04)^{-\frac{1}{2}}$.
18. $(.008)^{-\frac{1}{3}}$. **19.** $(144)^{-\frac{1}{2}}$. **20.** $(-32)^{\frac{2}{5}}$.
21. $(-8)^{-\frac{4}{3}}$. **22.** $(.01)^{-\frac{3}{2}}$. **23.** $(2.25)^{\frac{1}{2}}$.
24. $(1.44)^{-\frac{1}{2}}$. **25.** $(.125)^{-\frac{2}{3}}$.
26. $3^2\div 9^{\frac{1}{3}}=3^2\div 3^{\frac{2}{3}}=3^{2-\frac{2}{3}}=3^{\frac{4}{3}}$.
27. $4^2\div 2^3=2^4\div 2^3=2$.
28. $8^{\frac{1}{2}}\div 4^{\frac{1}{3}}=(2^3)^{\frac{1}{2}}\div (2^2)^{\frac{1}{3}}=2^{\frac{3}{2}}\div 2^{\frac{2}{3}}=2^{\frac{5}{6}}$.
29. $9^{\frac{2}{3}}\div 27^{\frac{1}{2}}$. **30.** $4^{\frac{2}{3}}\cdot 2^{\frac{2}{3}}$. **31.** $8^{\frac{3}{2}}\cdot 2^{-\frac{1}{2}}$.
32. $(3^{-2})^{-\frac{3}{2}}$. **33.** $(4^{\frac{1}{3}})^{-2}$. **34.** $[(8^{-2})^{-\frac{1}{2}}]^{\frac{2}{3}}$.

35. $27^{-\frac{2}{3}} \cdot 8^{-\frac{1}{3}} \div 4^{\frac{3}{2}} \cdot 6^2$.

36. $12^2 \div 3^3 \div 4 = (2^2 \cdot 3)^2 \div 3^3 \div 2^2 = 2^4 \cdot 3^2 \div 3^3 \div 2^2 = 2^2 \cdot 3^{-1} = \frac{4}{3}$.

37. $(18)^2 \div 6^2 \cdot 8^{\frac{2}{3}} \div 24$. **38.** $6^{-\frac{1}{2}} \cdot 4^2 \cdot 12^{\frac{1}{2}} \div 18^{-\frac{2}{3}} \div 6^{-\frac{2}{3}} \cdot 2^{\frac{2}{3}}$.

173. Simplification of radicals.

A radical is a root of a number indicated by a radical sign. If the required root can be found, the radical is said to be a **rational number**; if it cannot be found it is an **irrational number** or a **surd**. $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{a^2b^4}$, . . . are rational numbers. $\sqrt{5}$, $\sqrt[3]{10}$, $\sqrt[4]{a^3b^2}$, . . . are irrational numbers or surds. (See § 111.)

The number over the radical sign indicates the **order** of the radical. When no number is written, the square root is indicated, and the radical is of the second order; $\sqrt{2}$ is of the second order, $\sqrt[3]{2}$ is of the third order, and $\sqrt[n]{2}$ is of the nth order.

The radicals treated in Chapter VII were nearly all of the second order and were simplified, evaluated, added, subtracted, multiplied, and divided. Fractional exponents will be found helpful in explaining these processes and in dealing with higher orders of radicals.

Radicals may be simplified in several ways.

I. By reducing the order.

II. By removing a rational factor.

III. By removing the denominator if the radical is a fraction.

Illustrative examples.

1. $\sqrt[4]{9} = \sqrt[4]{3^2} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \sqrt{3}$. **3.** $\sqrt{50} = \sqrt{2 \cdot 25} = \sqrt{25}\,\sqrt{2} = 5\sqrt{2}$.

2. $\sqrt[6]{125} = \sqrt[6]{5^3} = 5^{\frac{3}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$. **4.** $\sqrt[3]{40} = \sqrt[3]{5 \cdot 8} = \sqrt[3]{8}\,\sqrt[3]{5} = 2\sqrt[3]{5}$.

5. $\sqrt{\frac{2}{5}} = \sqrt{\frac{2 \cdot 5}{5 \cdot 5}} = \sqrt{\frac{10}{25}} = \sqrt{10 \cdot \frac{1}{25}} = \frac{1}{5}\sqrt{10}$.

6. $\sqrt[3]{\frac{3}{4a}} = \sqrt[3]{\frac{3 \cdot 2a^2}{4a \cdot 2a^2}} = \sqrt[3]{\frac{6a^2}{8a^3}} = \frac{1}{2a}\sqrt[3]{6a^2}$.

Illustrative examples 3, 4, 5, and 6 are worked under the following—

Rule. *In multiplication or division radicals of the same order may be placed under the same radical sign and their product or quotient taken; and, vice versa, a radical expression may be dissolved into radical factors of the same order.*

This law is apparent if the radicals are written as quantities with fractional exponents.

For $\sqrt[3]{a}\cdot\sqrt[3]{b}=(a^{\frac{1}{3}}\cdot b^{\frac{1}{3}})=(ab)^{\frac{1}{3}}=\sqrt[3]{ab}$

and $\sqrt[3]{ab}=(ab)^{\frac{1}{3}}=a^{\frac{1}{3}}\cdot b^{\frac{1}{3}}=\sqrt[3]{a}\cdot\sqrt[3]{b}.$

Exercise 167

Follow the illustrative examples in simplifying the following:

1. $\sqrt[6]{36}.$ 2. $\sqrt[4]{25}.$ 3. $\sqrt[6]{216}.$ 4. $\sqrt[9]{125}.$

5. $\sqrt{12}.$ 6. $\sqrt[4]{48}.$ 7. $\sqrt[3]{54}.$ 8. $\sqrt[3]{250}.$

9. $\sqrt[4]{16a^5}.$ 10. $\sqrt[3]{81a^4b^2}.$ 11. $\sqrt[3]{56x^3y^4}.$ 12. $\sqrt[3]{686}.$

13. $\sqrt[4]{128}.$ 14. $\sqrt[3]{1024}.$ 15. $\sqrt[4]{729}.$ 17. $\sqrt[6]{128}.$

18. $\sqrt[3]{144}.$ 19. $\sqrt{\frac{3}{5}}.$ 20. $\sqrt{\frac{a}{b}}.$ 21. $\sqrt[3]{\frac{a}{b}}.$

Ans. $\frac{1}{b}\sqrt[3]{ab^2}.$ 22. $\sqrt[3]{\frac{16}{9a}}.$ 23. $\sqrt[3]{\frac{1}{2}}.$

24. $\sqrt[4]{\frac{1}{2}}.$ 25. $\sqrt[5]{\frac{1}{2}}.$ 26. $\sqrt[3]{\frac{2}{25a}}.$

27. $2\sqrt[3]{\frac{54}{4a^2}}.$ 28. $ab\sqrt[4]{\frac{2}{a^3b^2}}.$ 29. $\frac{4}{3}\sqrt[3]{\frac{x^4}{16y^2}}.$

30. $\sqrt{\frac{1}{a+b}}.$ 31. $\sqrt[3]{\frac{a}{(a+b)^2}}.$ 32. $\sqrt[3]{\frac{8a^2}{m+n}}.$

33. $(x+y)\sqrt{\frac{x-y}{x+y}}.$ 34. $(a+b)\sqrt[3]{\frac{a-b}{(a+b)^2}}.$

35. $4\pi\sqrt{\frac{3}{2\pi}}=2\sqrt{6\pi}.$ 36. $\frac{s}{3}\sqrt{\frac{s}{4\pi}}.$ 37. $4\pi\sqrt[3]{\frac{9v^2}{16\pi^2}}.$

Exercise 168

First simplify and then evaluate correct to .001 each of the following expressions, using the tables following § 238 for all square and cube roots.

1. $\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}=\frac{1}{3}(2.446)=?$
2. $\sqrt[3]{\frac{2}{5}}$. 3. $\sqrt{\frac{4}{5}}$. 4. $\sqrt[3]{\frac{3}{4}}$. 5. $\sqrt[3]{\frac{2}{7}}$.
6. $2-\sqrt{3}$. 7. $4-\sqrt{5}$. 8. $\sqrt[3]{\frac{1}{12}}$.
9. $\sqrt{512}$. 10. $\sqrt[3]{432}$. 11. $\sqrt{128}$.
12. $\sqrt{320}$. 13. $\sqrt{700}$. 14. $\sqrt[3]{500}$.
15. $\sqrt[3]{2000}$. 16. $\sqrt{3}-\sqrt{2}$.

174. Addition and subtraction of radicals.

Since every radical may be written without a radical sign by the use of fractional exponents, evidently the same rules will apply in the addition or subtraction of radicals that apply to other algebraic forms. We know that $2a^2+3a^2=5a^2$ and that $3a^{\frac{1}{2}}+5a^{\frac{1}{2}}=8a^{\frac{1}{2}}$. $\therefore 3\sqrt{a}+5\sqrt{a}=8\sqrt{a}$.

Exercise 169

Simplify and collect all similar terms:

1. $\sqrt{24}-\sqrt{6}+\sqrt{150}=2\sqrt{6}-\sqrt{6}+5\sqrt{6}=6\sqrt{6}$.
2. $\sqrt{32}-\sqrt{18}-\sqrt{8}+\sqrt{50}$. 3. $\sqrt{27}-\sqrt{12}+\sqrt{75}$.
4. $\sqrt[3]{16}-\sqrt[3]{250}+\sqrt[3]{54}$. Ans. 0.
5. $\sqrt{54}+2\sqrt{24}-\sqrt{96}$.
6. $2\sqrt[3]{81}-\sqrt[3]{192}+3\sqrt[3]{24}$.
7. $4\sqrt{\frac{3}{4}}-8\sqrt{\frac{3}{16}}+\sqrt{24}$.
8. $5\sqrt{3}+2\sqrt{48}-5\sqrt{108}$.
9. $\sqrt[3]{48}+\sqrt[3]{162}-\sqrt[3]{384}$.
10. $\sqrt[3]{500}+\sqrt[3]{256}-\sqrt[3]{32}-\sqrt[3]{108}$.
11. $\sqrt[3]{40}+5\sqrt[3]{\frac{1}{25}}-4\sqrt[3]{320}-\sqrt[3]{\frac{5}{8}}-\sqrt[3]{135}$.
12. $2\sqrt{4a^3}+\sqrt{9a^3}+\sqrt{25a^3}-\sqrt{36a^3}$.

13. $2\sqrt{\frac{x}{y}}-\sqrt{x^3y}-2\sqrt{\frac{y}{x}}=\left(\frac{2}{y}-x-\frac{2}{x}\right)\sqrt{xy}$

or $\frac{(2x-x^2y-2y)}{xy}\sqrt{xy}.$

14. $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}-2\sqrt{ab}.$

15. $a\sqrt{(a^2-b^2)(a+b)}+2ab\sqrt{a-b}-b\sqrt{(a-b)^3}.$

175. Comparison of values.

The relative values of radicals of different orders may be determined by the aid of fractional exponents.

Illustrative examples.

1. Which is the greater, $\sqrt{3}$ or $\sqrt[3]{5}$?

Now $\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{3}{6}}=\sqrt[6]{3^3}=\sqrt[6]{27}$,

and $\sqrt[3]{5}=5^{\frac{1}{3}}=5^{\frac{2}{6}}=\sqrt[6]{5^2}=\sqrt[6]{25}$.

$\therefore$ $\sqrt{3}$ is greater than $\sqrt[3]{5}$.

2. Arrange in ascending order of magnitude $\sqrt[3]{3}$, $\sqrt[4]{5}$, and $\sqrt[6]{10}$.

Reducing to the 12th order (the L. C. M. of 3, 4, and 6),

$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}$,

$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}$,

and $\sqrt[6]{10}=10^{\frac{1}{6}}=10^{\frac{2}{12}}=\sqrt[12]{10^2}=\sqrt[12]{100}$.

$\therefore \sqrt[3]{3}<\sqrt[6]{10}<\sqrt[4]{5}$.

Let the student state the process in the form of a rule.

Exercise 170

1. Which is the greater, $\sqrt[3]{4}$ or $\sqrt[4]{6}$?
2. Compare $\sqrt{2}$ and $\sqrt[5]{6}$.
3. Compare $\sqrt{7}$ and $\sqrt[3]{18}$.
4. Compare $\sqrt[4]{7}$ and $\sqrt[6]{19}$.

Arrange in ascending order:

5. $\sqrt{7}$, $\sqrt[3]{20}$, and $\sqrt[4]{50}$.
6. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$, and $\sqrt[6]{10}$.
7. $2\sqrt{3}$, $3\sqrt[3]{2}$, and $\sqrt[6]{2880}$. **Hint.** $2\sqrt{3}=\sqrt{12}$.
8. $2\sqrt{3}$, $2\sqrt[3]{5}$, and $\sqrt[6]{1700}$.
9. $5\sqrt{2}$, $2\sqrt[3]{6}$, and $4\sqrt[4]{10}$.

176. Multiplication and division of radicals.

Radicals of the same order may be placed under the common index and their product or quotient taken. (See § 173.)

For $\sqrt[3]{3}\cdot\sqrt[3]{5}=3^{\frac{1}{3}}\cdot 5^{\frac{1}{3}}=(3\cdot 5)^{\frac{1}{3}}=\sqrt[3]{3\cdot 5}=\sqrt[3]{15}$

and $\sqrt[3]{12}\div\sqrt[3]{3}=(12)^{\frac{1}{3}}\div 3^{\frac{1}{3}}=(12\div 3)^{\frac{1}{3}}=\sqrt[3]{12\div 3}=\sqrt[3]{4}.$

By the use of fractional exponents, radicals of different orders may be transformed into equivalent radicals of a common order following the plan of the last paragraph and their product or quotient taken.

Illustrative examples.

1. $\sqrt{x}\cdot\sqrt[3]{x}=x^{\frac{1}{2}}\cdot x^{\frac{1}{3}}=x^{\frac{5}{6}}=\sqrt[6]{x^5}.$
2. $\sqrt[3]{5}\cdot\sqrt{2}=5^{\frac{1}{3}}\cdot 2^{\frac{1}{2}}=5^{\frac{2}{6}}\cdot 2^{\frac{3}{6}}=\sqrt[6]{5^2\cdot 2^3}=\sqrt[6]{200}.$
3. $\sqrt[3]{a^2}\div\sqrt{a}=a^{\frac{2}{3}}\div a^{\frac{1}{2}}=a^{\frac{1}{6}}=\sqrt[6]{a}.$
4. $\sqrt[3]{6}\div\sqrt[4]{3}=6^{\frac{1}{3}}\div 3^{\frac{1}{4}}=6^{\frac{4}{12}}\div 3^{\frac{3}{12}}$
   $=\sqrt[12]{6^4\div 3^3}=\sqrt[12]{1296\div 27}=\sqrt[12]{48}.$

Exercise 171

Perform the indicated operations in each of the following:

1. $\sqrt{a}\cdot\sqrt[3]{a^2}$. Ans. $\sqrt[6]{a^7}$, or $a\sqrt[6]{a}.$
2. $\sqrt{x}\cdot\sqrt[5]{x^3}.$
3. $\sqrt[3]{a^2}\div\sqrt[4]{a}.$
4. $3\sqrt[3]{a^2}\div\sqrt{a}.$
5. $2\sqrt[4]{a^3}\cdot\sqrt{a}.$
6. $2\sqrt[4]{a^3}\div\sqrt{a}.$
7. $(3\sqrt{a}\sqrt[3]{b})^2.$
8. $(7\sqrt[3]{2x}\sqrt{y})^7.$
9. $(\sqrt{x}-\sqrt{y})^2.$
10. $(\sqrt{x}+\sqrt{y})^3.$
11. $(\sqrt{x}-\sqrt[3]{y})^2.$
12. $(\sqrt{x}-\sqrt[3]{y})^3.$
13. $(2a^{\frac{1}{3}}-3b^{\frac{1}{2}})^2.$
14. $(a^{\frac{1}{3}}-b^{\frac{1}{2}})^3.$
15. $(x-\sqrt{xy}+y)(\sqrt{x}+\sqrt{y}).$
16. $(\sqrt{x^3}-\sqrt{y^3})(\sqrt{x^3}+\sqrt{y^3}).$
17. $(x-y)\div(\sqrt{x}-\sqrt{y}).$
18. $(x+y)\div(\sqrt[3]{x}+\sqrt[3]{y}).$
19. $(x-y)\div(\sqrt[4]{x}-\sqrt[4]{y})\div(\sqrt[4]{x}+\sqrt[4]{y}).$

177. Rationalizing factor. It is convenient to transform a fraction with a radical denominator into an equivalent fraction with a rational denominator. This is known as rationalizing the denominator and depends upon the finding of a **rationalizing multiplier** or **factor.** (See § 120.)

Exercise 172

What is the simplest rationalizing factor for each of the following? Check by actual multiplication.

1. $\sqrt{7}$. Ans. $\sqrt{7}$. **2.** $2\sqrt{5}$. Ans. $\sqrt{5}$.
3. $\sqrt[3]{4}$. Ans. $\sqrt[3]{2}$. **4.** $\sqrt[3]{5}$. **5.** $\sqrt[4]{8}$.
6. $\sqrt[4]{2}$. **7.** $\sqrt[5]{a^4}$. **8.** $\sqrt[3]{a^2b}$. **9.** $\sqrt[3]{2ab^2}$.
10. $\sqrt[4]{27a^2b}$. **11.** $2-\sqrt{3}$. Ans. $2+\sqrt{3}$.
12. $\sqrt{7}-\sqrt{5}$. **13.** $2\sqrt{3}-3\sqrt{2}$. **14.** $\sqrt{a}+\sqrt{b}$.
15. $\sqrt{5}-\sqrt{3}-\sqrt{2}$. Try $(\sqrt{5}+\sqrt{3}+\sqrt{2})\cdot\sqrt{6}$. Why?
16. $\sqrt{3}-\sqrt{2}-1$. **17.** $\sqrt{a}-\sqrt{b}+c$.

When rationalizing the denominator of a fraction, multiply both terms by the simplest rationalizing factor of the denominator.

Illustrative examples.

1. $\dfrac{1}{\sqrt{2}}=\dfrac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\dfrac{\sqrt{2}}{2}$.

2. $\dfrac{2}{\sqrt[3]{4}}=\dfrac{2\cdot\sqrt[3]{2}}{\sqrt[3]{4}\cdot\sqrt[3]{2}}=\dfrac{2\sqrt[3]{2}}{2}=\sqrt[3]{2}$.

3. $\dfrac{3}{\sqrt{3}-\sqrt{2}}=\dfrac{3(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\dfrac{3(\sqrt{3}+\sqrt{2})}{3-2}=3\sqrt{3}+3\sqrt{2}$.

Exercise 173

Transform into equivalent fractions with a rational denominator and simplify:

1. $\dfrac{2}{3\sqrt{2}}$. Ans. $\dfrac{\sqrt{2}}{3}$. **2.** $\dfrac{2}{\sqrt{2}}$. **3.** $\dfrac{2}{\sqrt[3]{2}}$.

4. $\dfrac{3}{\sqrt[3]{9}}$. **5.** $\dfrac{a^2}{\sqrt{a}}$. **6.** $\dfrac{2a}{\sqrt[3]{4a}}$. **7.** $\dfrac{2ab}{\sqrt[4]{8a^2b^3}}$.

8. $\frac{12}{\sqrt[3]{6}}$. 9. $\frac{7}{2\sqrt[3]{49}}$. 10. $\frac{10}{\sqrt{50}}$. 11. $\frac{15}{\sqrt[3]{40}}$.

12. $\frac{3}{\sqrt{2}-1}$. 13. $\frac{4}{\sqrt{3}-1}$. 14. $\frac{5}{\sqrt{7}+\sqrt{2}}$.

15. $\frac{\sqrt{2}-4}{3\sqrt{2}-2\sqrt{3}}$. 16. $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{2}-3\sqrt{3}}$.

17. $\frac{a-b}{\sqrt{a}-\sqrt{b}}$. 18. $\frac{a\sqrt{b}-b\sqrt{a}}{b\sqrt{b}-a\sqrt{a}}$.

19. $\frac{28}{2\sqrt{6}-2\sqrt{2}}$. 20. $\frac{5}{\sqrt{14}-3}$. 21. $\frac{2\sqrt{3}-3\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

22. $\frac{6}{\sqrt{3}+\sqrt{5}+\sqrt{2}}$. 23. $\frac{4}{\sqrt{3}+\sqrt{2}+1}$.

24. $\frac{ab}{\sqrt{a}+\sqrt{b}+\sqrt{a+b}}$.

178. Exponential equations.

Exercise 174

Solve each of the following for x:

1. $4^x=16$. (Since $16=4^2$, $\therefore 4^x=4^2$ and $x=2$.)
2. $3^x=27$. 3. $9^x=27$. $[(3^2)^x=3^3$ or $3^{2x}=3^3$. $\therefore 2x=3$ and $x=\frac{3}{2}$.$]$
4. $2^x=8$. 5. $4^x=8$. 6. $16^x=8$.
7. $27^x=9$. 8. $81^x=9$. 9. $25^x=\frac{1}{5}$.
10. $32^x=16$. 11. $8^x=\frac{1}{16}$. 12. $8^{x-2}=\frac{1}{16}$.
13. $(\frac{1}{3})^x=27$. 14. $3^{x-1}=27$. 15. $(\frac{1}{2})^{-x}=8$.
16. $x^{\frac{2}{3}}=4$. **Hint.** $(x^{\frac{1}{3}})^2=2^2 \therefore x^{\frac{1}{3}}=2$ and $x=8$.
17. $(\frac{1}{4})^x=16$. 18. $(\frac{1}{27})^x=81$.
19. $x^{\frac{1}{2}}=2$. 20. $x^{\frac{1}{3}}=9$. 21. $x^{\frac{1}{3}}=\frac{1}{4}$.
22. $x^{-\frac{2}{3}}=25$. Ans. $x=\frac{1}{125}$.
23. $x^{-\frac{1}{2}}=\frac{9}{4}$. 24. $x^{-\frac{3}{2}}=1000$.
25. $4a^{-\frac{5}{3}}=\frac{1}{8}$. **Suggestion.** $a^{-\frac{5}{3}}=\frac{1}{32}$.
26. $\frac{3}{5}a^{-\frac{2}{3}}=\frac{5}{3}$. 27. $\frac{27}{8}a^{-3}=125$.

179. A **binomial quadratic surd** is a binomial one or both of whose terms is a surd of the second degree. $2+\sqrt{3}$, $\sqrt{2}-3\sqrt{5}$, and $\sqrt{5}-w$ are binomial quadratic surds. The general or typeform is $a+b\sqrt{c}$.

It is possible to obtain the square root of certain binomial quadratic surds, as follows:

By multiplication, the value of $(\sqrt{2}+\sqrt{3})^2$ is found to be $5+2\sqrt{6}$. The value of $(\sqrt{a}+\sqrt{b})^2$ is $a+b+2\sqrt{ab}$ where the rational part, $a+b$, is the sum of the two radicands and the radical part $\sqrt{ab}$ has for its radicand the product of the two radicands of the given binomial.

Therefore $\sqrt{7-2\sqrt{10}}$ can be simplified if 7 is the sum of two factors of 10. These factors are evidently 5 and 2 and $\sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}$ or $\sqrt{2}-\sqrt{5}$.

Check by finding the value of $(\sqrt{5}-\sqrt{2})^2$ and also $(\sqrt{2}-\sqrt{5})^2$.

The process may be stated as follows:

To find the square root of a binomial quadratic surd, transform the surd term into the form $2\sqrt{m}$. If m has two factors, a and b, whose sum is the rational term of the binomial, the square root of the binomial is $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$ according to the sign of the surd term in the given binomial.

Illustrative examples.

1. $\sqrt{7+4\sqrt{3}}=\sqrt{7+2\sqrt{12}}=\sqrt{4}+\sqrt{3}$ or $2+\sqrt{3}$.
Check by squaring $2+\sqrt{3}$.

2. $\sqrt{9-4\sqrt{5}}=\sqrt{9-2\sqrt{20}}=\sqrt{5}-\sqrt{4}$ or $\sqrt{5}-2$.

(**Note.** $\sqrt{9-4\sqrt{5}}$ may be written $2-\sqrt{5}$ but $\sqrt{5}-2$ is the positive root and $2-\sqrt{5}$ is the negative root. Why?)

The root, $\sqrt{5}-2$, is the principal root.
For discussion of principal root, see § 171.

Exercise 175

Find the positive square root of each of the following:

1. $8-2\sqrt{15}$.
2. $8-2\sqrt{7}$.
3. $12-6\sqrt{3}$.
4. $11-2\sqrt{30}$.
5. $11-4\sqrt{6}$.
6. $9+2\sqrt{18}$.
7. $31-10\sqrt{6}$.
8. $a+b-2\sqrt{ab}$.
9. $2a+3b+2\sqrt{6ab}$.
10. $\frac{3}{2}-2\sqrt{\frac{1}{2}}$.
11. $9\frac{1}{3}-2\sqrt{3}$.
12. $a^2+b+2a\sqrt{b}$.

180. Imaginary numbers and the imaginary unit. The square root of a number has been defined as one of its two equal factors. Since a negative number cannot have two equal factors, its square root cannot be found.

The product of $(+2)\ (+2)$ and also of $(-2)\ (-2)$ is $+4$, therefore the square root of $+4$ (written $\pm\sqrt{4}$) may be either $+2$ or -2. Similarly, $(+\sqrt{3})\ (+\sqrt{3})$ and $(-\sqrt{3})\ (-\sqrt{3})$ both equal 3, therefore the square root of 3 may be either $+\sqrt{3}$ or $-\sqrt{3}$.

But -4 is not the product of two equal factors therefore its square root can only be expressed, as $\sqrt{-4}$. Similarly for the square root of -3 and of all other negative numbers. The fourth root of $+16$ is either $+2$ or -2 (prove by multiplication), but there are no four equal factors whose product is -16 and the fourth root of -16 can only be expressed, as $\sqrt[4]{-16}$.

Such indicated even roots of negative numbers are known as **imaginary numbers,** or simply **imaginaries.** So far in our study of mathematics, all numbers have been real numbers.

Such numbers as $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-7}$, and $\sqrt{-30}$ may be written as the product of two factors, that is $\sqrt{-2}=\sqrt{2}\cdot\sqrt{-1}$, $\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}$, $\sqrt{7}=\sqrt{7}\cdot\sqrt{-1}$, and $\sqrt{-30}=\sqrt{30}\cdot\sqrt{-1}$. One of these factors, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{7}$, $\sqrt{30}$, . . . , is a real number and the other, $\sqrt{-1}$, is a new number which has been conveniently called the **imaginary unit.** Every indicated square root of a negative number may be separated into two factors, one of which is the imaginary unit.

The study of imaginary numbers is largely the interpretation and application of this imaginary unit.

The letter i is frequently used as the symbol for the imaginary unit. $3\sqrt{-1}$ is written $3i$, $\sqrt{-2}$ is written $i\sqrt{2}$, and $\sqrt{-3}$ is written $i\sqrt{3}$.

Now by the definition of a square root we know that $(\sqrt{-1})^2=-1$, or $i^2=-1$. Similarly,
$(\sqrt{-1})^3=(\sqrt{-1})^2\cdot(\sqrt{-1})=-\sqrt{-1}$, or $i^3=i^2\cdot i=-i$.

Also $(\sqrt{-1})^4=[(\sqrt{-1})^2]^2=(-1)^2=+1$,
or $i^4=(i^2)^2=(-1)^2=+1$.

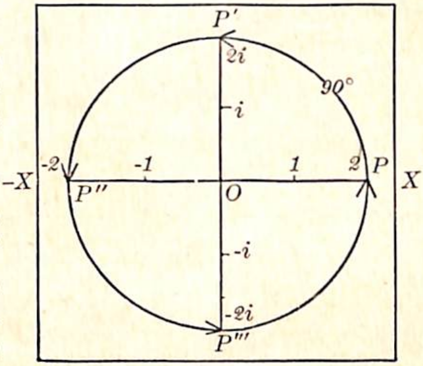
Or, in tabular form:

$$\begin{aligned}\sqrt{-1}&=i\\(\sqrt{-1})^2&=i^2=-1\\(\sqrt{-1})^3&=i^3=-\sqrt{-1},\text{ or }-i\\(\sqrt{-1})^4&=i^4=+1.\end{aligned}$$

181. Interpretation of the imaginary unit.

It is from an application of this table to the number scale that mathematicians have obtained a reasonable and very useful interpretation of the meaning of the imaginary unit.

On the number scale XX′, let P be taken at the point $+2$, that is, let the line-segment OP be $+2$ units long. Similarly, let P″ be taken at the point -2 and the line-segment OP″ be -2 units long.



The line-segment OP may be made to coincide with the line-segment OP″ by turning it on the plane about O as a pivot through an angle of 180°. But the number, $+2$, may be converted into the number -2 by multiplying $+2$ twice by $\sqrt{-1}$, or by i^2.

If the angle of 180° is divided into two equal angles each 90°, then, since multiplication by $(\sqrt{-1})^2$ is equivalent to turning the segment through an angle of 180°, multiplication by $\sqrt{-1}$ may be thought of as geometrically equivalent to turning a segment counter-clockwise through an angle of 90°. And we may define the imaginary unit as an operator that rotates a segment about a point on a plane through an angle of 90° with each application. This interpretation of the meaning of the imaginary unit expands our idea of number from that of position on a directed line to position on a plane. Its application will be found very helpful in many places in higher mathematics.

The sum or the difference of a real number and an imaginary number is called a **complex number,** such as $2+\sqrt{-3}$ and $\sqrt{2}+\sqrt{-5}$.

Its general type may be written $a+b\sqrt{-1}$, or $a+bi$, where a and b are real numbers. (For interpretation, see Chap. XXII.)

182. Numbers that have $\sqrt{-1}$, or i, as a factor may be treated as real numbers have been treated provided care is used in simplifying all powers of $\sqrt{-1}$, or i, above the first under the typeform $(\sqrt{-1})^2 \equiv -1$.

Illustrative examples.

1. $\sqrt{-12}+\sqrt{-27}-3\sqrt{-\frac{1}{3}}=\sqrt{12}\ \sqrt{-1}+\sqrt{27}\ \sqrt{-1}-3\sqrt{\frac{1}{3}}\ \sqrt{-1}$ $=2i\sqrt{3}+3i\sqrt{3}-i\sqrt{3}=4i\sqrt{3}.$

2. $(\sqrt{-18})\ (2\sqrt{-12})=(3i\sqrt{2})\ (4i\sqrt{3})=12i^2\sqrt{6}=-12\sqrt{6}.$

3. $(\sqrt{a}+\sqrt{-b})\ (\sqrt{a}-\sqrt{-b})=(\sqrt{a}+i\sqrt{b})\ (\sqrt{a}-i\sqrt{b})=a-i^2b=$ $a+b.$

4. $\dfrac{3}{\sqrt{2}-\sqrt{-1}}=\dfrac{3(\sqrt{2}+i)}{(\sqrt{2}-i)\ (\sqrt{2}+i)}=\dfrac{3(\sqrt{2}+i)}{2-i^2}=\dfrac{3(\sqrt{2}+i)}{3}=\sqrt{2}+i.$

Exercise 176

Follow the plan of the illustrative examples in simplifying the following:

1. $\sqrt{-16}+\sqrt{-9}-\sqrt{-25}$. Ans. $2i$.
2. $\sqrt{-27}+\sqrt{-75}-\sqrt{8}$. Ans. $8i\sqrt{3}-2\sqrt{2}$.
3. $2\sqrt{-27}-3\sqrt{-25}+4\sqrt{-9}$.
4. $5\sqrt{-81a^2}-8\sqrt{-36a^2}$.
5. $2\sqrt{-8}-4\sqrt{-12}+3\sqrt{-18}+5\sqrt{-27}$.
6. $\sqrt{-50}-\sqrt{-75}+\sqrt{-72}+\sqrt{-98}$.
7. $4\sqrt{-96}+2\sqrt{-54}+\sqrt{-150}-\sqrt{-125}$.
8. $2\sqrt{-a^3}+\sqrt{-4a}+\sqrt{-9a^5}$.
9. $3\sqrt{-x^3}-\sqrt{-49a^2x}+\sqrt{-36a^4x^5}$.
10. $(3\sqrt{-1})(\sqrt{-9})$.
11. $(2\sqrt{-3})(3\sqrt{-2})$.
12. $(\sqrt{-18})(\sqrt{-9})(\sqrt{-3})^2$.
13. $(-\sqrt{-12})(\sqrt{-18})$.
14. $(\sqrt{-21a})(\sqrt{-14b})(\sqrt{-6c})$.
15. $(\sqrt{-6a^3})(\sqrt{-3a})(\sqrt{-2b^3})$.
16. $(a\sqrt{-1})(b\sqrt{-1})(c\sqrt{-1})(d\sqrt{-1})$.
17. $(3+\sqrt{-2})(3-\sqrt{-2})$.
18. $(2\sqrt{a}-3\sqrt{-b})(2\sqrt{a}+3\sqrt{-b})$.
19. $(\sqrt{5}+2\sqrt{-3})^2$.
20. $(\sqrt{3}+\sqrt{-2})^3$.
21. $(\sqrt{3}+2\sqrt{-2})(2\sqrt{3}-\sqrt{-2})(10-3\sqrt{-6})$.
22. $(\sqrt{-2}+\sqrt{-3}-\sqrt{-5})^2$.
23. $(\sqrt{-a}+\sqrt{-b}+\sqrt{-c}-\sqrt{-9})^2$.
24. $\dfrac{6}{\sqrt{-3}}$.
25. $\dfrac{2a}{\sqrt{-a}}$.
26. $\dfrac{5}{\sqrt{3}-\sqrt{-2}}$.
27. $\dfrac{ab}{a\sqrt{-b}-b\sqrt{-a}}$.
28. $\dfrac{6(\sqrt{5}+\sqrt{-7})}{\sqrt{5}-\sqrt{-7}}$.

Find one square root for each of the following:

29. $1+2\sqrt{-6}$. Ans. $\sqrt{3}+\sqrt{-2}$.
30. $2-2\sqrt{-15}$. Ans. $\sqrt{5}-\sqrt{-3}$.

31. $4-6\sqrt{-5}$. **32.** $1-12\sqrt{-2}$.

33. If $x=-1+\sqrt{-1}$, find the value of x^2+2x+7. Ans. 5.

34. If $x=\dfrac{3-\sqrt{-3}}{2}$, find the value of x^2-3x+3.

35. Does $x=\dfrac{-1+\sqrt{-7}}{4}$ satisfy $2x^2+x+1=0$?

Exercise 177. Review

Find one square root for each of the following polynomials (See § 109):

1. $x^4-2x^3-3x^2+4x+4$.

2. $4a^4-12a^3b+a^2b^2+12ab^3+4b^4$.

3. $4a^2+12ab+9b^2-20ac-30bc+25c^2$.

4. $n^4-2n^3+2n^2-n+\frac{1}{4}$.

5. $\dfrac{9a^4}{16}-\dfrac{3a^3}{4}+\dfrac{5a^2}{8}-\dfrac{a}{4}+\dfrac{1}{16}$.

6. $9a^{\frac{4}{3}}-12a^{\frac{3}{3}}-8a^{\frac{2}{3}}+8a^{\frac{1}{3}}+4$.

7. $9x^{-\frac{4}{3}}-12x^{-1}-14x^{-\frac{2}{3}}+12x^{-\frac{1}{3}}+9$.

8. $4n^{-2}-12n^{-1}+17-12n+4n^2$.

Find one square root for each of the following numbers (See § 110):

9. 50625. **10.** 139129. **11.** 91385.29.
12. 18.2329. **13.** 65124900. **14.** 2579236.

Find one square root for each of the following, correct to .001:

15. 39. **16.** 2. **17.** 7. **18.** 29.3.
19. 338.105. **20.** .89.

21. Multiply $2n^{-2}-3n^{-1}-2$ by $3n^{-2}+n^{-1}+4$.

22. Multiply $2a^{\frac{2}{3}}-3a^{\frac{1}{3}}+5$ by $a^{\frac{2}{3}}-2a^{\frac{1}{3}}-2$.

23. Divide $6x-7x^{\frac{4}{5}}+6x^{\frac{3}{5}}-12x^{\frac{2}{5}}+4x^{\frac{1}{5}}+3$ by $3x^{\frac{2}{5}}-2x^{\frac{1}{5}}-1$. Check with $x=32$.

Express the following in simplest radical form:

24. $2\sqrt{12}-\sqrt{75}+6\sqrt{\frac{1}{3}}$.

25. $\sqrt{20}-4\sqrt{\frac{1}{5}}-5^{-\frac{1}{2}}+5^{\frac{3}{2}}$.

26. $\sqrt{12}-9\sqrt{\frac{1}{3}}+8^{-\frac{1}{3}}\cdot 6^0.$

27. $\dfrac{3}{\sqrt{5}-\sqrt{2}}+\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}.$

28. $\sqrt{24}+\dfrac{3}{\sqrt{6}}-\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}.$

29. $(\frac{4}{5})^{\frac{1}{2}}+13\sqrt{\frac{1}{5}}+\sqrt[4]{\dfrac{25}{81}}.$

30. $\dfrac{\sqrt{-5}+\sqrt{-3}}{\sqrt{-5}-\sqrt{-3}}-\dfrac{\sqrt{-5}-\sqrt{-3}}{\sqrt{-5}+\sqrt{-3}}.$

Express the following with positive exponents:

31. $\dfrac{m^{-1}-n^{-1}}{m^{-2}-n^{-2}}\div\dfrac{m^{-1}n^{-1}}{(m+n)^{-1}}.$

32. $\dfrac{ab^{-1}+a^{-1}b+1}{a^{-3}-b^{-3}}\cdot\dfrac{a^{-2}-b^{-2}}{a^{-1}b^{-1}}.$

33. $2\cdot 8^{\frac{2}{3}}-\sqrt{2}(18)^{\frac{1}{2}}-2\cdot 5^0+\left(\dfrac{a+b}{ab}\right)^{-1}-\dfrac{1}{a^{-1}+b^{-1}}.$

Simplify the following and approximate the results, correct to .001:

34. $\sqrt{8}+5\sqrt{\frac{1}{2}}-18^{\frac{1}{2}}+20^{-1}\cdot 50^{\frac{3}{2}}.$

35. $\dfrac{6}{\sqrt{3}}-9\sqrt{\frac{1}{3}}+12^{\frac{1}{2}}-25^0.$

36. $8^{-\frac{2}{3}}+\dfrac{1}{16^{\frac{3}{4}}}-25^{\frac{1}{2}}+8^{-\frac{1}{2}}.$

Find one square root for each of the following:

37. $9-6\sqrt{2}.$

38. $2m+2\sqrt{m^2-n^2}.$

39. $2+2\sqrt{-15}.$

40. $a^2-b-2a\sqrt{-b}.$

41. If $x=-2-\sqrt{3}$, find the value of x^2+4x+7.

42. Does $x=\dfrac{3-\sqrt{-7}}{2}$ satisfy $x^2-3x+4=0$?

CHAPTER XV

LOGARITHMS

183. Introduction. Any positive number may be written as a power of some other positive number. For instance, $4=2^2$, $8=2^3$, $\frac{1}{9}=3^{-2}$. If a is greater than 0 and $a^x=b$, x is said to be the **logarithm** of b to the **base** a, and b is the **anti-logarithm** of x to the base a. Written compactly the first statement is $\log_a b=x$, which is to be read "the logarithm of b to the base a is x."

Exercise 178

Find x by inspection in each of the following:

1. $\log_2 8=x$.
2. $\log_3 27=x$.
3. $\log_2 \frac{1}{8}=x$.
4. $\log_3 \frac{1}{9}=x$.
5. $\log_4 8=x$. Ans. $x=\frac{3}{2}$.
6. $\log_{10} 100=x$.
7. $\log_4 \frac{1}{8}=x$.
8. $\log_{10} .1=x$. Ans. $x=-1$.
9. $\log_{10} .01=x$.
10. $\log_{10} .001=x$.
11. $\log_9 27=x$.
12. $\log_8 4=x$.
13. $\log_{\frac{1}{5}} 25=x$.
14. $\log_{16} 8=x$.

Since 6 is between 4 and 8, 6 is between the second and the third powers of 2, or it is more than 2^2, and less than 2^3. Written in the form used above we have $\log_2 6=2. \ . \ . \ .$, or the logarithm of 6 to the base 2 is 2+ some decimal. When the value of this decimal is computed it is found to be approximately .5849. $\therefore 6=2^{2.5849}$, or $\log_2 6=2.5849$.

184. The integral part of the logarithm, or 2, is called the **characteristic,** and the decimal part, .5849, is called the **mantissa.** In like manner any positive number may be written as a power of 2 or of any other positive number except 1.

Common logarithms have 10 for their base and the accompanying tables give the mantissas of the logarithms to the base 10 of all the numbers from 100 to 999 approximately correct to the fourth place of decimals.

To find from the tables the logarithm of such a number as 457, we note first that 457 is more than 100 and less than 1000, therefore it is more than 10^2. For the decimal part, or the mantissa, we look in the tables down the first column at the left for the first two significant figures, 45, and then along their line across the page to the column headed 7, the third significant figure, where we find 6599. This is the approximate decimal value of the mantissa and $457 = 10^{2.6599}$, or $\log_{10} 457 = 2.6599$.

Since 10 is the most commonly used base for all logarithmic tables, it is not customary to name the base unless some other number is to be used, that is, $\log 457 = 2.6599$ will be the form and is to be read "logarithm of 457 is 2.6599."

Exercise 179

Find the logarithm for each of the following:

1. 247. (Write your answer $\log 247 = 2.3927$.)
2. 299. **3.** 425. **4.** 755. **5.** 333. **6.** 129. **7.** 999.
8. 500. **9.** 852. **10.** 102. **11.** 771. **12.** 101.

185. The characteristic.

Since $45.7 = \frac{1}{10}$ of 457, or $10^{-1}(457)$, therefore $45.7 = 10^{-1}\,10^{2.6599}$, or $10^{1.6599}$, that is, $\log 45.7 = 1.6599$.

Also $4.57 = \frac{1}{100}$ of 457, or $10^{-2}(457)$, therefore $\log 4.57 = 0.6599$.

Now $4570 = 10^1(457)$, therefore $\log 4570 = 3.6599$,
and $45700 = 10^2(457)$, therefore $\log 45700 = 4.6599$.

Evidently the characteristic depends upon the place of the decimal point in the number, the logarithm of which is to be determined.

The method for determining the characteristic of the logarithm of a number will appear from a study of the following table:

$1000 = 10^3$,	or $\log 1000 = 3$,		$\log 4570 = 3.6599$.
$100 = 10^2$,	or $\log 100 = 2$,		$\log 457 = 2.6599$.
$10 = 10^1$,	or $\log 10 = 1$,		$\log 45.7 = 1.6599$.
$1 = 10^0$,	or $\log 1 = 0$,		$\log 4.57 = 0.6599$.
$.1 = 10^{-1}$,	or $\log .1 = -1$,	or $\bar{1}$,	$\log .457 = \bar{1}.6599$.
$.01 = 10^{-2}$,	or $\log .01 = -2$,	or $\bar{2}$,	$\log .0457 = \bar{2}.6599$.
$.001 = 10^{-3}$,	or $\log .001 = -3$,	or $\bar{3}$,	$\log .00457 = \bar{3}.6599$.

It should be noticed that the mantissa is the same in the logarithms of 4570, 457, 45.7, etc., because 4570 is the same multiple of 1000 that 457 is of 100, 45.7 of 10, and 4.57 of 1.

The mantissa is always positive but the characteristic is negative when the number is less than unity.

Evidently the characteristic of the logarithm of a number with integral places is positive and one less than the number of integral places. The characteristic of the logarithm of a decimal number is negative and determined by counting the decimal point and the number of ciphers to the right of the decimal point and between it and the first significant figure, or it is determined by the decimal position of the first significant figure.

There are two ways of writing a negative characteristic: (1) as in the preceding table with a minus sign over the characteristic, or, (2) 10 is added to and subtracted from the logarithm, as follows:

$$\text{Log } .457 = \bar{1}.6599, \text{ or } 9.6599 - 10$$
$$\text{Log } .0457 = \bar{2}.6599, \text{ or } 8.6599 - 10$$
$$\text{Log } .00457 = \bar{3}.6599, \text{ or } 7.6599 - 10.$$

We will use the second way throughout this chapter.

Exercise 180

Name at sight the characteristic of each of the following:

1. 234. **2.** 23.4. **3.** 2.34. **4.** .234. **5.** .0234.
6. .000234. **7.** 999999. **8.** 77777.7. **9.** 100000000.
10. 1000.001. **11.** 10.000007. **12.** .000000001.

Find the mantissa from the tables and write the logarithm of each of the following:

13. .0999. Ans. log .0999 = 8.9996 − 10.
14. 57700. **15.** 4590. **16.** .00577. **17.** 3580000.
18. 93000000. **19.** 5280. **20.** .00004. **21.** .000000435.

186. To find an antilogarithm. If the logarithm of a number is known, it is only necessary to reverse the process in using the tables to locate the number.

What number has 2.5119 for its logarithm, or what is the antilogarithm of 2.5119?

A search through the tables of mantissas locates 5119 as the mantissa of 325 and, since the characteristic is 2, the antilogarithm of 2.5119 is 325.

Similarly, log 63300 = 4.8014 antilog 4.8014 = 63360.

Exercise 181

Find the antilogarithm of each of the following:

1. 1.2201. **2.** 5.3010. **3.** 3.4871. **4.** 8.8779 − 10.
5. 7.6464 − 10. **6.** 9.8779 − 10. **7.** 0.3010. **8.** 7.9991.

187. The approximate logarithm of a number of four figures may be found from these tables in the following manner:

Given the number 2086. We find the mantissa of 208 which is 3181 and of 209 which is 3201. The difference between these two mantissas is 20. Now 2086 is $\frac{6}{10}$ of the interval between 2080 and 2090, therefore its logarithm should have the mantissa 3181 + 6/10 of 20, or 3181 + 12, making it 3193. Therefore log 2086 = 3.3193.

The logarithm of 457.6 is 2.6605.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Exercise 182

Find the approximate logarithm of each of the following:

1. 44.45. **2.** 234.7. **3.** 125800. **4.** .002755.
5. 3.141. **6.** 9990. **7.** 7926. **8.** 4.188. **9.** .3125.
10. .0007224. **11.** 6666. **12.** 1728. **13.** 1.732.

Study

The fourth figure of an antilogarithm may be found approximately by a method similar to that used in the preceding paragraph.

Given that log $x=3.3575$, find the number x. When we compare the mantissa 3575 with the table we find that it lies between 3560 and 3579 which are the mantissas of the logarithms of 227 and 228, respectively.

Evidently 3575 is the mantissa of a number between 227 and 228 and 3.3575 is the logarithm of a number between 2270 and 2280. Since 3579 is 19 more than 3560 and 3575 is 15 more than 3560, 3.3575 is the logarithm of a number greater than 2270 by 15/19 of 10 which is approximately 8. Therefore the antilogarithm of 3.3575 is 2278. Check this by finding log 2278.

188. To find the antilogarithm when the given mantissa is not in the table, find from the table the next lower mantissa with the corresponding three figures of the antilogarithm, subtract this mantissa of the table from the given mantissa, and divide this difference by the difference between this tabular mantissa and the next higher mantissa of the table. This quotient gives the fourth figure of the antilogarithm. The characteristic indicates the position of the decimal point.

Exercise 183

Find the antilogarithm of each of the following:

1. 2.5535. **2.** 3.6586. **3.** 1.9304. **4.** 0.4971.
5. 9.8239−10. **6.** 3.2375. **7.** 0.2386. **8.** 1.5000.
9. 6.8660−10. **10.** 2.9942. **11.** 9.0999−10.
12. 3.7368. **13.** 7.4747−10. **14.** 6.5999

189. Applications of logarithms. Since a logarithm is an exponent, its use furnishes a short cut to the solution of the following arithmetical operations:

I. Finding the product of several numbers, under the law of exponents expressed by the typeform $a^m \cdot a^n = a^{m+n}$.

II. Finding the quotient of one number divided by another under the typeform $a^m \div a^n = a^{m-n}$.

III. Finding a required power of a given number under the typeform $(a^m)^n = a^{mn}$.

IV. Finding a required root of a given number under the typeform $\sqrt[m]{a^n} = a^{\frac{n}{m}}$.

It is to be understood that the accompanying tables will give results approximately correct only to the fourth significant figure. If more than four places are required, more extensive tables may be used and the answer obtained correct to any reasonable number of places. The processes, however, are identical with those followed in using the accompanying four place tables.

Illustrative examples.

1. Multiply 277 by 383.

Add their logarithms. (277 times $383 = 10^{2.4425}$ times $10^{2.5832} = 10^{5.0257} = 106100$.)

log 277 = 2.4425
log 383 = 2.5832
antilog of 5.0257 = 106100. Ans.

2. Find 4.55·.00773·25.6.

Add their logarithms.

log 4.55 = 0.6580
log .00775 = 7.8893 − 10
log 25.6 = 1.4082
antilog of 9.9555 − 10 = .9026. Ans.

3. Divide 378 by 28.9.

Find the difference of the logarithms.

log 378 = 2.5775
log 28.9 = 1.4609
antilog of 1.1166 = 13.08. Ans.

4. Divide .0444 by 64.5.

$$\begin{aligned}\log .0444 &= 8.6474-10\\ \log 64.5 &= 1.8096\\ \hline \text{antilog of } 6.8378-10 &= .0006883. \quad \text{Ans.}\end{aligned}$$

5. Divide 24.77 by 9765.

$$\begin{aligned}\log 24.77 = 1.3940 &= 11.3940-10\\ \log 9765 &= 3.9897\\ \hline \text{antilog of } \quad 7.4043-10 &= .002537. \quad \text{Ans.}\end{aligned}$$

6. Find the value of $(2.75)^5$.

Multiply the logarithm of 2.75 by 5. Why?

$$5(\log 2.75) = 5(.4393) = 2.1965$$

$$\text{antilog of } 2.1965 = 157.2. \quad \text{Ans.}$$

7. Find the value of $\sqrt[5]{243}$.

Divide the logarithm of 243 by 5. Why?

$$\frac{1}{5}(\log 243) = \frac{1}{5}(2.3855) = 0.4771$$

$$\text{antilog } 0.4771 = 3. \quad \text{Ans.}$$

8. Find the cube root of .02.

$$\frac{1}{3}(\log .02) = \frac{1}{3}(8.3010-10) = \frac{1}{3}(28.3010-30) = 9.4337-10$$

$$\text{antilog } 9.4337-10 = .2714. \quad \text{Ans.}$$

Note. When logarithms are used in finding a root of a decimal number, as in Example 8, it is necessary that the negative part of the logarithm of the required root be divisible by the index of the root. This is easily accomplished if it is noticed that $8.3010-10 = 28.3010-30$.

Exercise 184

Solve each of the following by the use of logarithms:

1. 365×24.
2. $84.75 \times 3.67 \times .024$.
3. $93000000 \div 186000$.
4. $\dfrac{4.775 \times 36}{28.45}$.
5. $\sqrt[3]{3}$.
6. $\sqrt{7}$.
7. $(24)^4$.
8. $(.085)^3$.
9. $\sqrt[3]{.001728}$.
10. $\sqrt[5]{12}$.
11. $\sqrt[7]{875.9}$.
12. $(.125)^7$.
13. $(1.44)^3$.

14. $\sqrt[3]{.007}$. Hint. $\frac{1}{3}(7.8451-10)=\frac{1}{3}(27.8451-30)$.

15. $\dfrac{392\times\sqrt[3]{1728}}{(4)^3\times\sqrt{24.3}}$.

16. $\dfrac{\sqrt{45}\cdot\sqrt[3]{92}\cdot\sqrt[4]{15}}{(2.32)^2\cdot\sqrt{11}}$.

17. $\dfrac{25\cdot\sqrt{42}\cdot\sqrt[3]{121}}{\sqrt{.0125}\cdot\sqrt[3]{215}}$.

18. $\dfrac{\sqrt{2}\cdot\sqrt[4]{4}}{\sqrt[3]{3}\cdot\sqrt[5]{5}}$.

19. $\sqrt[3]{\dfrac{27\times36\times48}{347\times.8975}}$.

20. $\sqrt{\dfrac{96.75\times6.475}{234.7\times47.2}}$.

190. Since (1) $a^m\cdot a^n\equiv a^{m+n}$, (2) $a^m\div a^n\equiv a^{m-n}$, (3) $(a^m)^n\equiv a^{mn}$, and (4) $\sqrt[n]{a^m}\equiv a^{\frac{m}{n}}$, certain logarithmic identities can be established.

Illustrative example.

Log $ab=\log a+\log b$.

Proof. Let $a=10^m$ and $b=10^n$.

Then $ab=10^m\cdot10^n=10^{m+n}$ $\therefore \log ab=m+n=\log a+\log b$.

Exercise 185

Using the plan of the illustrative example, prove each of the following identities:

1. Log $\frac{a}{b}=\log a-\log b$.
2. Log $a^3=3\log a$.
3. Log $a^m=m\log a$.
4. Log $\sqrt[n]{a^m}=\frac{m}{n}\log a$.
5. Log $abc=\log a+\log b+\log c$.
6. Log $\frac{ab}{c}=\log a+\log b-\log c$.
7. Log $\frac{ab^2}{c^3}=\log a+2\log b-3\log c$.
8. Log $\sqrt[3]{\frac{a^2b}{c}}=\frac{1}{3}(2\log a+\log b-\log c)$.
9. Log $\sqrt{a^2-b^2}=\frac{1}{2}[\log(a+b)+\log(a-b)]$.

10. $\text{Log}\sqrt[5]{\frac{a^3b^2}{\sqrt{c}}}=\frac{1}{5}\left[3\log a+2\log b-\frac{1}{2}\log c\right]$.

Complete the following logarithmic identities:

11. $\text{Log}\ \pi r^2=?$ 12. $\text{Log}\ \frac{4}{3}\pi r^3=?$ 13. $\text{Log}\sqrt{\frac{S}{4\pi}}=?$

14. $\text{Log}\sqrt{s(s-a)(s-b)(s-c)}=?$ 15. $\text{Log}\sqrt[3]{\frac{a^2-b^2}{c^2}}$.

Exercise 186. Problems

Solve by logarithms:

1. If a ray of light travels 186000 miles per second and it is 93,000,000 miles from the sun to the earth, how long does it take a ray of light to reach the earth from the sun?

2. The distance from the earth to the stars is measured by astronomers in light-years (the distance a ray of light travels in one year). How many miles is it to a star that is estimated as 8 light-years from the earth?

3. A certain bright star is estimated to be 300 light years from the earth. How many miles away is it?

4. What will $5000 amount to if placed at 6% for 5 years, interest to be compounded annually?

Hint. One dollar at 6% for 5 years would amount to $(1.06)^5$.

5. What would be the amount, if the interest in the preceding problem were compounded semi-annually?

Hint. Use $(1.03)^{10}$.

6. What will $50 amount to in 20 years at 7%, compounded annually?

7. In what time will $1 double itself at 6%, compounded annually?

Hint. $(1.06)^x=2$, therefore $x(\log 1.06)=\log 2$ and $x=\frac{\log 2}{\log 1.06}$.

8. In what time will $1 double itself at 6%, compounded semi-annually?

9. In what time will $500 amount to $750 at 6%, interest compounded annually?

10. What sum of money put at 6%, interest compounded semi-annually, will amount to \$10000 in 5 years?

11. What sum of money at 6%, interest compounded annually, will amount to \$10000 in 5 years?

12. The area of a circle is expressed by the formula $A = \pi r^2$. What is the area of a circle whose radius is 25 inches?

13. How many inches in the radius of a circle whose area is 1 square foot (144 sq. in.)?

14. The area of the equilateral triangle is expressed by the formula $A = \frac{s^2}{4}\sqrt{3}$. How many square inches in the area of an equilateral triangle whose side is one foot?

15. A regular hexagon is inscribed in a circle whose radius is 18 inches. What is the difference between their areas?

16. The apothem of a regular hexagon is $\frac{s}{2}\sqrt{3}$ where s is a side. Find the area of a regular hexagon whose apothem is 12 inches.

17. Find the radius of the circle circumscribed about a square whose side is 12 inches.

18. Find the radius of the circle circumscribed about a regular hexagon whose area is 144 square inches.

19. Find the apothem of the regular hexagon whose area is 144 square inches.

20. The area of the surface of a sphere is expressed by the formula $S = 4\pi r^2$. What is the area of the surface of a ball 10 inches in diameter?

21. How many square miles are on the surface of the earth assuming it to be a sphere whose radius is 3960 miles?

22. How many square miles are on the surface of the sun if its radius is approximately 433000 miles?

23. How many inches in the radius of the sphere whose area is 144 square inches?

24. The volume of a sphere is expressed by the formula $V=\frac{4}{3}\pi r^3$. What is the volume of a sphere whose radius is 12 inches?

25. What is the radius of the sphere whose volume is 1 cubic foot?

26. How many cubic miles are in the volume of the earth?

27. How many cubic miles are in the volume of the sun?

28. A cubic mile of water weighs how many tons, if one cubic foot of water weighs $62\frac{1}{2}$ pounds?

29. The volumes of two spheres have the ratio of the cubes of their radii. The volume of the sun is how many times the volume of the earth?

30. A cubic foot of lead will make how many spherical shot $\frac{1}{4}$ of an inch in diameter?

31. The area of a triangle whose sides are a, b, and c is expressed by the formula $A=\sqrt{s(s-a)(s-b)(s-c)}$ where $2s=a+b+c$. What is the area of the triangle whose sides are 17, 23, and 30 inches, respectively?

32. How many acres are in a triangular field whose sides are 755.5, 909.5, and 1325 feet, respectively?

33. What is the side of an equilateral triangle whose area is 1 square foot (144 square inches)?

34. Find the weight of a spherical ball of lead whose diameter is 8 in.

Note. Specific gravity is defined as the ratio of the weight of a body to the weight of an equal volume of water. For instance, the specific gravity of lead is 11.36 which means that a cubic foot of lead weighs 11.36 times 62.5 pounds, the weight of one cubic foot of water.

35. What would be the increase in the weight of the lead ball in No. 34 if 2 inches were added to its diameter?

36. Find the volume of a hemispherical cap of granite whose radius is 9 inches.

37. What is the weight of a cubic foot of gold, if its specific gravity is 19.27?

Third quarter Exams ended here March 17, 1927.

CHAPTER XVI

QUADRATIC EQUATIONS

191. Definitions. A **quadratic equation** is an equation of the second degree with respect to the unknown. If it contains no term of the first degree, it is called a **pure quadratic,** or an **incomplete quadratic.** If it contains terms of both the first and the second degree it is called an **affected quadratic** equation.

The equation $x^2 = 16$ is a pure quadratic. Likewise $x^2-5x=6$ and $x^2=3x$ are affected quadratic equations.

If an affected quadratic has one term that does not contain the unknown, it is called a **complete quadratic** equation. $x^2-5x=6$ is a complete quadratic as well as an affected quadratic.

Every quadratic equation in x can be reduced to the form $ax^2+bx+c=0$, where a may have any known value other than 0, and b and c may have any known values whatsoever.

As, for example, $(3-2x)(3+2x)=5-6x$ becomes $4x^2-6x-4=0$ when put in the form $ax^2+bx+c=0$.

192. Review. The student is already familiar with many of the processes of quadratic equations. However, a thorough review of the four types of problems in the following exercise will make the later work of the chapter easier to master.

Exercise 187

Solve the following equations by factoring:

1. $x^2-25=0.$ **2.** $a^2-7a+12=0.$
3. $x^2-11x+24=0.$ **4.** $4x^2-49=0.$
5. $3a^2+11a+8=0.$ **6.** $5n^2-2n=7.$
7. $x^3-81x=0.$ **8.** $13a=6a^2+6.$
9. $20=6n^2-7n.$ **10.** $x^3-3x^2-54x=0.$
11. $3a^3=5a^2-2a.$ **12.** $x^4-5x^2+4=0.$

Write the equations whose roots are the following:

13. $4, 5.$ **14.** $-6, 3.$ **15.** $-2, -7.$
16. $-7, 5.$ **17.** $a, b.$ **18.** $-b, c.$
19. $2b, 3c.$ **20.** $-3a, 4b.$ **21.** $\frac{2}{3}, -\frac{1}{2}.$

Solution. $(x-\frac{2}{3})(x+\frac{1}{2})=0$ becomes $x^2-\frac{1}{6}x-\frac{1}{3}=0$ and, clearing of fractions, gives $6x^2-x-2=0.$

22. $2, -\frac{1}{2}.$ **23.** $-3, \frac{2}{3}.$ **24.** $-\frac{2}{3}, -\frac{3}{2}.$
25. $x, -\dfrac{x}{3}.$ **26.** $\dfrac{n}{3}, \dfrac{3n}{2}.$ **27.** $2a, \dfrac{1}{2a}.$

Reduce the following pure quadratics to the form $x^2=a$ and then find the two roots by taking the square root of each member, as $x=\pm\sqrt{a}$:

28. $x^2=25.$ **29.** $4x^2=9.$ **30.** $x^2=7.$
31. $4x^2=15.$ **32.** $9x^2=4a^2b^2.$ **33.** $7x^2=8a.$

Supply the missing terms in the following trinomial squares:

34. $a^2+8a+?$ **35.** $n^2-10n+?$
36. $9a^2-6a+?$ **37.** $x^2+?+36a^2.$
38. $4n^2-?+9.$ **39.** $x^2+x+?$
40. $4a^2-?+\frac{1}{4}b^2.$ **41.** $9x^2-?+\frac{1}{4}.$
42. $9a^2-3a+?$ **43.** $\frac{4}{9}a^2-?+16y^2.$

193. Solution of a quadratic equation by completing the square.

First method. Complete the left member of the equation into a perfect trinomial square with the coefficient of x^2 positive 1.

Illustrative examples.

1. $x^2+8x=20.$

Solution.

$x^2+8x+16=36.$ (Completing the square.)

$x+4=\pm 6.$ (Extracting the square root of both members.)

$x=\pm 6-4,\quad \therefore x=2$, and $x=-10.$

Both roots check.

2. $3x^2+7x=-3.$

Solution.

$x^2+\frac{7}{3}x=-1.$ (Making the coefficient of x^2 positive 1.)

$$x^2+\frac{7}{3}x+\frac{49}{36}=\frac{13}{36}.$$

$$x+\frac{7}{6}=\pm\frac{3.606}{6}.$$

$$x=\pm\frac{3.606}{6}-\frac{7}{6}\ \therefore x=-.566 \text{ and } x=-1.768.$$

These roots are approximate. Why? Check for both.

Note. When the coefficient of x^2 is $+1$, the addition of the square of one-half the coefficient of x to both members transforms the left member into a perfect trinomial square.

Exercise 188

Solve the following quadratic equations by completing the square. If the roots are irrational simplify and reduce them to decimal form. Irrational roots may be checked in their radical form.

1. $n^2+6n=40.$ **2.** $a^2-12a=-20.$
3. $x^2-7x=18.$ **4.** $n^2+3n=40.$
5. $3x^2+8x+5=0.$ **6.** $5a^2+7a=6.$
7. $6a^2-11a-10=0.$ **8.** $x^2+6x=-7.$
9. $a^2-10a=-7.$ **10.** $3n^2+4n=9.$
11. $4=3a^2-2a.$ **12.** $5x^2+12x-17=0.$
13. $x^2+3x=5.$ **14.** $2x^2=5x+3.$
15. $n^2+2n=3.$ **16.** $8=3x^2-5x.$

Second method. (This method may be omitted at the discretion of the teacher and the problems of the exercise solved by the first method or by the formula.) Fractions may be avoided in completing the square by using some other number than 1 for the coefficient of x, as in the following:

Illustrative examples.

1. $3x^2+7x=-2$.

Solution.

$36x^2+84x=-24$. (Multiplying both members of the equation by 4 times the coefficient of x^2.)

$36x^2+84x+49=25$. (Adding to both members the square of the coefficient of x.)

$6x+7=\pm 5$. (Extracting the square root.)

$\therefore x=-\frac{1}{3}$ and $x=-2$.

2. $3x^2-4x=11$.

Solution.

$9x^2-12x=33$. (Multiplying both members by the coefficient of x^2.)

$9x^2-12x+4=37$. (Adding to both members the square of one-half the coefficient of x.)

$3x-2=\pm 6.083$. $\therefore x=2.694$ and $x=-1.361$.

Note. To avoid fractions in completing the square reduce the equation to the form $ax^2+bx=-c$ and multiply both members by $4a$ if b is an odd number, or by a if b is an even number. To complete the square, divide the coefficient of x by twice the square root of the coefficient of x^2 and add the square of the result to both members.

Exercise 189

Solve the following quadratic equations by the second method:

1. $2a^2+3a=9$.

2. $4n^2-3n=7$.

3. $2x^2-x=6$.

4. $6x^2-x-2=0$.

5. $3n^2-4n=7$.

6. $3n^2-4n=8$.

7. $a^2+3a=4$.

8. $5n^2+2n=3$.

9. $8=2x^2+3x$.

10. $(2-x)(2+x)=5x$.

194. Solution of quadratic equations by the formula.

Since every quadratic equation can be reduced to the form $ax^2+bx+c=0$, where a, b, and c are the respective coefficients of x^2, x, and x^0 (or the term not containing x), it is evident that if we solve this equation for x, we can use the resulting roots as a **formula** to obtain the roots of any quadratic equation.

Solution.

$$ax^2+bx+c=0.$$

$$ax^2+bx=-c.$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}. \qquad \therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

$$\text{or } x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Illustrative examples.

1. $2x^2-x=6.$

Solution.

$2x^2-x-6=0.$ a is 2, b is -1, and c is -6.

Substituting in the formula. $x=\frac{1\pm\sqrt{1+48}}{4}=\frac{1\pm 7}{4}.$

$\therefore x=2$ and $x=-\frac{3}{2}.$ Both roots check.

2. $2x^2+3x+7=0.$

Solution. a is 2, b is 3, and c is 7.

$$x=\frac{-3\pm\sqrt{9-56}}{4}. \qquad x=\frac{-3+\sqrt{-47}}{4} \text{ and } x=\frac{-3-\sqrt{-47}}{4}.$$

Check. (See § 182.)

Exercise 190

Solve the following quadratic equations by the formula:

1. $2x^2-3x+1=0.$ **2.** $3x^2-2x-1=0.$

3. $3x^2+x-2=0.$ **4.** $3x^2+8x+4=0.$

5. $7a^2-a-8=0.$ **6.** $2n^2-n-10=0.$

7. $2x^2+5x=3.$ **8.** $5a^2-a=6.$

9. $2a^2+3a=4.$ **10.** $3a^2+5a=7.$

11. $4a^2+5a+1=0.$ **12.** $2n^2+7n+6=0.$

13. $3a^2+6a+4=0$. Ans. $a=\frac{-3\pm\sqrt{-3}}{3}$.

14. $x^2+3x+4=0$. **15.** $2n^2-3n+2=0$.

16. $5a^2+5a+1=0$. **17.** $2x^2+x+\frac{1}{2}=0$.

18. $x^3-1=0$. Ans. $x=1$, $x=\frac{-1\pm\sqrt{-3}}{2}$.

Suggestion. By factoring $x-1=0$ and $x^2+x+1=0$.

19. $x^3+1=0$. **20.** $x^3-8=0$. **21.** $x^3+8=0$.

22. $x^4+x^2+1=0$.

Suggestion. $(x^2+x+1)(x^2-x+1)=0$.

23. $x^4-3x^2+9=0$. **24.** $x^6-1=0$.

Solve the following equations for x:

25. $2n^2x^2+nx-3=0$. **26.** $3x^2-5ax=2a^2$.

27. $2x^2-ax-10a^2=0$. **28.** $ax^2+bx-a+b=0$.

29. $6abx^2-4a^2x-9b^2x+6ab=0$.

Suggestion. $6abx^2-(4a^2+9b^2)x+6ab=0$.

30. $acx^2-3bcx+2anx=6bn$.

31. $2x^2-\frac{11bx}{3a}-\frac{10b^2}{3a^2}=0$. **32.** $\frac{ab-bx}{2x-a}=a-x$.

33. $\frac{2n+x}{2n-x}-\frac{2x-n}{2x+n}=\frac{8}{3}$. **34.** $\frac{2x-a}{a+2x}+\frac{a+2x}{2x-a}=\frac{5}{2}$.

195. Character of the roots of a quadratic equation. The student will observe in the quadratic formula that—

(1) If $b^2-4ac=0$, both roots of the quadratic equation are $\frac{-b}{2a}$, and are therefore real and equal.

(2) If b^2-4ac is a perfect square, the roots are real, rational, and unequal.

(3) If b^2-4ac is a positive number but not a perfect square, the roots are real, irrational, and unequal.

(4) If b^2-4ac is a negative number, the roots are imaginary.

The expression b^2-4ac is called the **discriminant** of the quadratic equation.

An outline of numbers will be convenient in the work of the exercise that follows.

- Numbers
  - Real
    - Rational
    - Irrational
  - Imaginary
    - Simple imaginary
    - Complex

Exercise 191

In each of the following equations determine the value of b^2-4ac *and, without solving, classify the roots:*

1. $x^2-6x+8=0$. **2.** $4x^2-20x+25=0$.

3. $3x^2+5x-8=0$. **4.** $2x^2+3x-7=0$.

5. $x^2+x+1=0$. **6.** $2x^2+5x+4=0$.

7. In the equation $x^2-10x+k=0$, what must be the value of k in order that the roots may be equal?

8. Determine k in the equation $9x^2-kx+4=0$ so that the roots may be equal.

9. Find the greatest value m may have in $3x^2+2x+m=0$ so that the roots may be real.

10. Find the least value m may have in $2x^2+mx+2=0$ so that the roots may be real.

11. Find two factors of 1 whose sum is 3.

Suggestion. Let x and $\frac{1}{x}$ represent the factors.

12. Find two factors of 3 whose sum is 1.

13. Find the least value m may have in $3x^2+2x+m=0$ so that the roots may be complex.

14. One root of $x^3-2x^2-x-6=0$ is 3. Find the other roots.

Suggestion. One factor of x^3-2x^2-x-6 is $x-3$. Why?

15. One root of $x^3+3x^2+5x+6=0$ is -2. Find the other roots.

196. Relation of the roots and the coefficients of a quadratic equation. By dividing both members of a quadratic equation by the coefficient of x^2 and collecting terms in the left member, any quadratic equation may be put in the form $x^2+px+q=0$. This is called the p-form of the quadratic equation to distinguish it from $ax^2+bx+c=0$ which is called the a-form. It is evident that the a-form may be changed to the p-form and that $p=\frac{b}{a}$ and $q=\frac{c}{a}$.

Solving $x^2+px+q=0$ by the formula gives

$$x=\frac{-p+\sqrt{p^2-4q}}{2} \text{ and } x=\frac{-p-\sqrt{p^2-4q}}{2}.$$

The sum of these two roots is evidently $-p$ and the product is $\frac{(-p)^2-(\sqrt{p^2-4q})^2}{4}$ or q.

The two facts that the sum of the roots is $-p$ and the product q will be helpful in

(a) forming a quadratic equation when its roots are given,

(b) checking the roots resulting from the solution of a quadratic by determining if their sum is $-p$ and their product q.

Illustrative examples.

1. Write the equation whose roots are $5-\sqrt{7}$ and $5+\sqrt{7}$.

Solution.

Adding, $-p=10$ or $p=-10$.

Multiplying $q=18$.

Therefore $x^2-10x+18=0$ is the equation.

2. Does $x=\frac{4\pm\sqrt{-5}}{3}$ satisfy $3x^2-8x+7=0$?

Solution. The equation in the p-form is $x^2-\frac{8}{3}x+\frac{7}{3}=0$. Whence $p=-\frac{8}{3}$ and $q=\frac{7}{3}$. $\frac{4+\sqrt{-5}}{3}+\frac{4-\sqrt{-5}}{3}=\frac{8}{3}=-p$ and $\frac{4+\sqrt{-5}}{3}\cdot\frac{4-\sqrt{-5}}{3}=\frac{16-(-5)}{9}=\frac{21}{9}=\frac{7}{3}=q$. Ans. Yes.

Exercise 192

Solve each of 1–10 in two ways. Write the equations whose roots are:

1. $5, -3$.
2. $-7, -4$.
3. $3, -\frac{2}{3}$.
4. $\frac{8}{3}, \frac{3}{2}$.
5. $3-\sqrt{7}, 3+\sqrt{7}$.
6. $\frac{7\pm\sqrt{5}}{2}$.
7. $\frac{5\pm\sqrt{3}}{3}$.
8. $\frac{4\pm\sqrt{-3}}{2}$.
9. $\frac{3\pm\sqrt{-2}}{2}$.
10. $\frac{-5\pm\sqrt{-3}}{2}$.

11. Does $x=\frac{3\pm\sqrt{-7}}{4}$ satisfy $2x^2-3x+2=0$?

12. Does $x=\frac{1+3\sqrt{-1}}{2}$ satisfy $2x^2-2x+5=0$?

13. One root of $x^2+8x+k=0$ is 2. Find the other root and the value of k.

Note. No. 13 can be done in two ways: $x=2$ must satisfy the equation, and the sum of the roots is -8. Why?

14. One root of $3x^2-5x+n=0$ is $\frac{2}{3}$. Find the other root and n.

15. One root of $x^2-kx+20=0$ is 4. Find the other root and k.

16. Find m and n in the equation $x^2+mx+n=0$ so that the roots may be -5 and -8.

17. One root of $x^2-11x+k=0$ is 3 more than the other. Find the roots and k.

Suggestion. What is the sum of the roots?
How can you find two numbers whose sum is 11 when one is 3 more than the other?

18. One of the roots of $x^2-9x+k=0$ is twice the other. Find the roots and k.

197. Equivalent equations. Equations that involve the same unknown number and have the same roots are called **equivalent equations.** As, for example, $\frac{x+4}{x}=\frac{x-2}{x-3}$ and $2x=8$ are equivalent equations, for $x=4$, satisfies both and no other value of x satisfies either. In the solution of a quadratic equation (or of any other type) it is necessary that the equations obtained in the successive steps of the solution be equivalent, each to the original and therefore to one another, otherwise roots may be introduced or roots may be lost.

198. Extraneous roots and lost roots. When a root is obtained in the solution of an equation that does not check in the original equation, it is called an **extraneous root.**

A study of the following **illustrative examples** will serve to show how roots may be introduced or lost in the solution of equations.

1. The equation $2x-3=x+2$ is satisfied by $x=5$.

If each member of this equation is multiplied by $x-3$, we get $2x^2-9x+9=x^2-x-6$, which becomes $x^2-8x+15=0$, or $(x-5)(x-3)=0$. Therefore $x=3$ and $x=5$. But $x=3$ does not check in the original equation and is an extraneous root.

2. The equation $2x^2-10x+12=x^2-5x+6$ is satisfied by $x=2$ and $x=3$.

If each member is divided by $x-3$, we get $2x-4=x-2$, which becomes $x-2=0$. Therefore $x=2$. The root $x=3$ has been lost.

3. Solve the equation $\frac{3x}{x-3}+x+1=\frac{9}{x-3}$.

Clearing of fractions gives $3x+x^2-2x-3=9$, or $x^2+x-12=0$ which is satisfied by $x=3$ and $x=-4$.

But $x=3$ does not check in the original equation and is extraneous. The root $x=3$ was introduced when the original equation was cleared of fractions by multiplying each member by $x-3$. If the solution had been as follows no extraneous root would have entered:

$$\frac{3x}{x-3}+x+1=\frac{9}{x-3}. \qquad \frac{3x}{x-3}-\frac{9}{x-3}+x+1=0.$$

$$\frac{3x-9}{x-3}+x+1=0. \qquad 3+x+1=0 \text{ and } x=-4.$$

In the solution of any type of equations it is best to keep in mind (1) that checking will detect an extraneous root, (2) that no factor containing the unknown can be removed without removing a root.

199. Irrational equations resulting in quadratics. Many irrational equations reduce to the quadratic form in the process of solution. Often one of the resulting roots is extraneous and care must be used in checking both.

Illustrative examples.

1. $\sqrt{3x+1}=x-1$.

Solution. $3x+1=x^2-2x+1$. (Squaring both members.)

Whence $x^2-5x=0$ and $x(x-5)=0$, or $x=5$ and $x=0$.

Checking. If $x=5$, $\sqrt{15+1}=5-1$ and $4=4$, which checks.

If $x=0$, $\sqrt{0+1}=0-1$ and $\sqrt{1}=-1$ which does not check.

2. $\sqrt{x+5}+\sqrt{x}=\sqrt{6x+1}$.

Squaring both members, $x+5+2\sqrt{x^2+5x}+x=6x+1$.

Combining, $2\sqrt{x^2+5x}=4x-4$.

Dividing by 2, $\sqrt{x^2+5x}=2x-2$.

Squaring, $x^2+5x=4x^2-8x+4$. Whence $3x^2-13x+4=0$.

Therefore $(3x-1)(x-4)=0$ and $x=4$, $x=\frac{1}{3}$.

Checking. If $x=4$, $\sqrt{9}+\sqrt{4}=\sqrt{25}$, and $3+2=5$. Checks.

If $x=\frac{1}{3}$, $\sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{3}$, and $\frac{4}{3}\sqrt{3}+\frac{1}{3}\sqrt{3}=\sqrt{3}$. Does not check.

Exercise 193

Solve and check the following irrational equations:

1. $\sqrt{6x-5}=2x-5$. **2.** $\sqrt{2a^2+2a+1}=a+2$.

3. $\sqrt{3n+3}=4-\sqrt{n-1}$. **4.** $\sqrt{2x-5}-\sqrt{x-6}=2$.

5. $\sqrt{n+4}+\sqrt{2n-1}=6$.

6. $\sqrt{m-4}=\sqrt{4m+5}-\sqrt{3m+1}$.

7. $\sqrt{2a-3}-\sqrt{3a-4}=\sqrt{a-1}$.

8. $\sqrt{x^2+4\sqrt{7x+15}}=x+2$.

9. $\sqrt[3]{x^3-5x^2-2x+2}=x-1$.

10. $\sqrt{25+4x}-\sqrt{8+x}-\sqrt{2x+9}=0$.

11. $\sqrt{25+4x}+\sqrt{8+x}-\sqrt{2x+9}=0$.

12. $\sqrt{4x-6n}=\sqrt{3x+2n}-\sqrt{2n}$. (Solve for x.)

200. Solution of equations of higher degree. The complete solution of all types of equations of higher degree than the second is beyond the realm of elementary algebra, and even the processes of advanced mathematics fail in many cases. However, there are some types that can be solved by applying the principles already learned.

Illustrative examples.

Case I. Equations that may be reduced by factoring.

1. $x^3-64=0$.

Factoring $(x-4)(x^2+4x+16)=0$.
Solving $x-4=0$ and $x^2+4x+16=0$,
$\therefore x=4$ and $x=-2\pm 2\sqrt{-3}$.

Note. All roots should be checked by substituting in the original equation. It will be observed that any equation of third or higher degree can be solved if it can be reduced by factoring to two or more equations that are linear or quadratic.

Case II. Equations that are quadratic in form with respect to some expression containing the unknown.

2. $x^4+3x^2+1=0$.

This is quadratic with respect to x^2.

Then $x^2=\dfrac{-3\pm\sqrt{5}}{2}$.

Whence $x=\pm\sqrt{\dfrac{-3\pm\sqrt{5}}{2}}=\pm\frac{1}{2}\sqrt{-6\pm 2\sqrt{5}}$.

3. $(x^2+5x)^2-5(x^2+5x)=6$.

Solution.

Put $x^2+5x=n$. Then $n^2-5n=6$.
Whence $n=6$ and $n=-1$.
Then $x^2+5x=6$. $\therefore x=-6$, and $x=1$.

Also $x^2+5x=-1$. $\therefore x=\dfrac{-5\pm\sqrt{21}}{2}$.

4. $x^2+2x-6\sqrt{x^2+2x+10}=-15$.

Solution.

$x^2+2x+10-6\sqrt{x^2+2x+10}=-5$. (Adding 10 to each number.)
Put $\sqrt{x^2+2x+10}=n$. Then $n^2-6n=-5$.
Whence $n=1$ or $n=5$.
Then $\sqrt{x^2+2x+10}=1$. $\therefore x^2+2x+10=1$.
Whence $x=-1\pm 2\sqrt{-2}$.
$\sqrt{x^2+2x+10}=5$. $\therefore x^2+2x+10=25$.
Whence $x=3$ and $x=-5$. Check.

Exercise 194

Find all the roots of the following equations and check:

1. $x^3-27=0$. **2.** $(x-3)(2x^2+7x+2)=0$.

3. $(n^2-n+1)(n^2+n+1)=0$.

4. $x^4-9x^2+20=0$. **5.** $a^6-64=0$.

6. $x-x^{\frac{1}{2}}=20$. **7.** $3a^{\frac{2}{3}}+4a^{\frac{1}{3}}=4$.

8. $\sqrt{x}-3\sqrt[4]{x}=40$. **9.** $2x+3-5\sqrt{2x+3}=-6$.

10. $x^3-4x^{\frac{3}{2}}+3=0$. **11.** $(a-2)^3-3(a-2)^{\frac{3}{2}}=40$.

12. $a^2+3a+3\sqrt{a^2+3a}=10$. **13.** $x^4-7x^2+1=0$.

Note. No. 13 can be solved in two ways. First by substituting n for x^2 in the equation, second, by factoring. Solve by both methods and show that the roots obtained by one method agree with those obtained by the other method.

14. $x^4+3x^2+4=0$. **15.** $4x^4-x^2+4=0$.

16. $x^{-2}-5x^{-1}+6=0$. **17.** $6x^{-2}+6=13x^{-1}$.

18. $2\left(x+\frac{1}{x}\right)^2-9\left(x+\frac{1}{x}\right)+10=0$.

19. $\sqrt{\frac{3-5x}{3-x}}+\sqrt{\frac{3-x}{3-5x}}=\frac{5}{2}$. **20.** $\frac{17}{4}-\frac{x+3}{3x-7}=\frac{3x-7}{x+3}$.

201. Use of the quadratic equation. When a quadratic equation is used in the solution of a problem, care is necessary not only in checking the roots of the equation but also in determining whether both of these roots satisfy the conditions imposed by the problem. As, for instance, the length of a rectangle cannot be a negative number.

Exercise 195. Problems

1. Find two consecutive numbers whose product is 90.

2. Find two consecutive odd numbers the sum of whose squares is 130.

3. Find three consecutive numbers the sum of whose squares is 110.

4. Separate the number 20 into two parts, the sum of whose squares shall be 202.

5. The area of a rectangular field is 180 square rods and its perimeter is 56 rods. Find its length and width.

6. A number of boys engaged a motor boat for a trip for which they agreed to pay $40. One of the boys was not able to go and each of the others paid $2 more than he expected to pay. How many boys went on the trip?

7. The sum of the reciprocals of two consecutive even numbers is $\frac{5}{12}$. Find the numbers.

Note. The reciprocal of an integral number is one divided by the number.

8. Separate the number 12 into two parts such that one part shall equal the square of the other.

9. Separate the number k into two parts such that one part shall equal the square of the other.

10. Find four consecutive numbers such that the sum of the squares of the first and second exceeds the product of the third and fourth by 5.

11. A field that is 8 rods longer than it is wide has an area of 8 acres. Find its dimensions.

12. Separate 72 into two parts such that the square of one part is equal to the other part.

13. One side of a right triangle is 24 inches and the hypotenuse is 6 inches more than twice the length of the other side. Find the unknown side and the hypotenuse.

14. Two chords intersect within a circle. The segments of one chord are 6 inches and 7 inches and the total length of the other chord is 17 inches. Find the segments of the second chord.

Note. If two chords intersect within a circle, the product of the segments of one equals the product of the segments of the other.

15. A number of boys bought a boat for $36. One of them failed to pay, so each of the others paid 50 cents more than he had agreed to pay. Find the number of boys.

16. The number of diagonals that can be drawn in a polygon of n sides is expressed by the formula $d = \frac{1}{2}n(n-3)$. If in a certain polygon 20 diagonals can be drawn, find the number of sides. Find the number of sides in a polygon that has d diagonals.

17. The number of degrees in the three angles of a triangle are represented by x^2, $\frac{1}{2}x^2+3$, and $2x+7$. Find x and each angle.

18. The values of the four angles of a quadrilateral are expressed in degrees as follows: x^2, x^2+3x, $7x+12$, and $10x+6$. Find x and all the angles.

19. Find the altitude and area of an equilateral triangle whose side is 12 inches.

20. A ladder 20 feet long leans against the wall of a building and makes an angle of 60 degrees with the ground. Find the distance of the foot of the ladder from the wall and the height of the wall to the point where the ladder touches it.

21. The base of a triangle is 12 inches and the angles adjacent to the base are 60 and 45 degrees, respectively. Find the other two sides of the triangle.

22. The number of feet, d, an object will fall in t seconds when it has an initial velocity downward, v, is expressed by the formula $d = 16t^2 + vt$. If a ball is started downward from the top of a tower 180 feet high at the rate of 12 feet per second, in how many seconds will it reach the ground?

23. An object falling from the top of a building passes a window 120 feet from the ground at the rate of 20 feet per second. Find the number of seconds required for the object to reach the ground after passing the window.

24. The area of a trapezoid whose altitude is h and bases b and b_1, is expressed by the formula $A = \frac{1}{2}h(b+b_1)$. One base of a certain trapezoid is 4 inches more than the altitude and the other equals the altitude. Find the altitude and bases if the area is 48 square inches.

25. A rectangular piece of tin, 6 inches longer than it is wide, is made into an open box by cutting a 3 inch square from each corner and turning up the sides. If the volume of the box is 216 cubic inches, find the dimensions of the original piece of tin.

26. A certain number exceeds 4 times its square root by 12. Find the number.

27. A man bought a number of sheep for $250. Four of the sheep died and he sold the others for $2.50 per head more than he paid and received $240. How many sheep did he buy?

28. A landscape gardener has a flower bed that is 12 yards wide and 18 yards long. He wishes to double the area of the flower bed by increasing the width and length the same amount. Find the number of yards the dimensions must be increased.

29. A farmer starts to cut a field of grain, 50 rods by 60 rods, by driving his reaper round and round the field. Find the width of the strip that is cut when two-thirds of the field remains uncut.

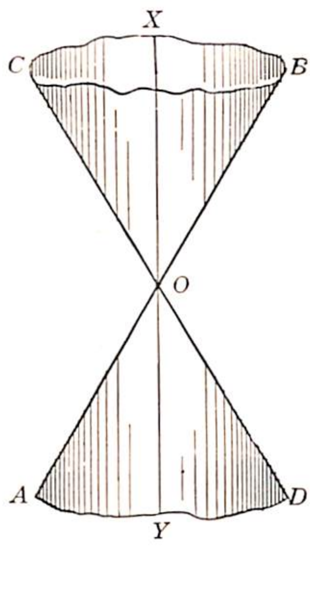
CHAPTER XVII

QUADRATIC EQUATIONS THAT INVOLVE TWO VARIABLES

202. Quadratic equations as conic sections. If the graph of any quadratic equation in two variables be drawn, we have a curved line belonging to one of the four groups of curves known as **conic sections.** These curves are the **circle,** the **parabola,** the **hyperbola,** and the **ellipse.** The definitions of these are as follows:

A **circle** is the **locus** (or path) of a point that is always at a given distance from a fixed point.

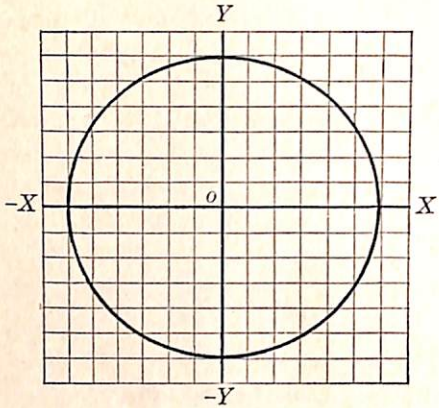
A **parabola** is the locus of a point whose distance from a fixed point, called the **focus,** is always equal to its distance from a fixed line, called the **directrix.**



A **hyperbola** is the locus of a point the difference of whose distances from two fixed points, called the **foci,** is always equal to a constant distance.

An **ellipse** is the locus of a point, the sum of whose distances from two fixed points, called the **foci,** is always equal to a constant distance.

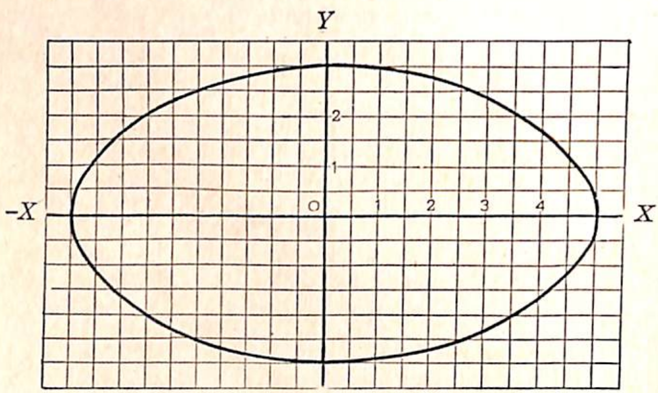
203. Conic sections. Suppose the straight lines AB and CD intersect at O and that XY is the



Circle $x^2 + y^2 = 36$

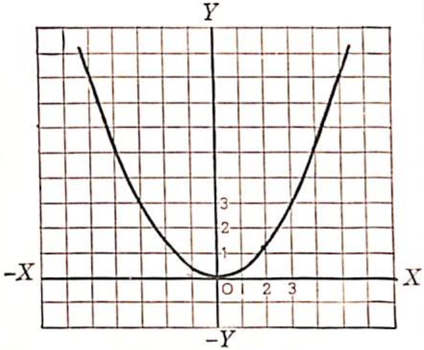
bisector of two opposite angles. Now if the whole figure be revolved about XY as an axis, the lines AB and CD will generate (or form) a **conical surface.** This surface will have two parts, or **nappes,** meeting point to point. The conical surface will be indefinite, or unlimited, in extent, since AB and CD are lines of unlimited length. Next, suppose the conical surface to be intersected by a plane surface. If the plane is perpendicular to XY, the intersection will be a circle.

If the plane is oblique to XY and cuts completely through one nappe, the intersection will be an ellipse.



Ellipse $9x^2 + 25y^2 = 225$

If the plane is parallel to one of the lines AB or CD in any of its positions, it will cut one nappe only and the

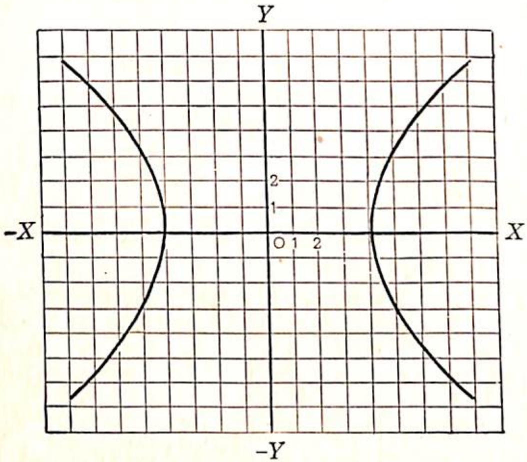


Parabola $x^2 = 3y$

intersection will be a parabola. If the plane cuts both of the nappes, the intersection will be the double open curve, or hyperbola.

Note. If a quadratic equation of two unknowns can be factored into two linear equations, the graph in this case will be a pair of straight lines, or the respective graphs of the two linear equations.

204. Graphing a quadratic equation in two variables. In graphing a quadratic equation in two variables, it is best to solve the equation for one variable in terms of the other and substitute numbers, both positive and negative, for the second variable until a sufficient number of sets of values has been found to locate the curve.



Hyperbola $x^2 - y^2 = 16$

Illustrative example. Graph $x^2 - y^2 = 16$.

Solution.

Solving for x, gives $x = \pm\sqrt{16 + y^2}$.

Substituting values for y, gives the following sets:

When $y =$	0,	2,	-2,	3,	-3,	4,	-4,
then $x =$	± 4,	± 4.5,	± 4.5,	± 5,	± 5,	± 5.6,	± 5.6.

Notice that there are two sets of values for each value of y, for when $y = 2$, then $x = +4.5$ and $x = -4.5$.

See the table of square roots following Chapter XXII.

Exercise 196

Graph the following equations:

1. $y=\frac{12}{x}$.
2. $y=\frac{12}{x}-3$.
3. $x^2-y^2=25$.
4. $2x^2-5y^2=50$.
5. $x^2+y^2=20$.
6. $2x^2+3y^2=30$.
7. $(x+3)^2+y^2=36$.
8. $(x+2)^2+(y-3)^2=25$.
9. $y=x^2$.
10. $y=x^2+3$.
11. $y=x^2-6x+9$.
12. $y=x^2-6x-3$.
13. $x=y^2$.
14. $x=y^2+7$.
15. $y=x^2+2x$.
16. $y=2x^2+4x$.

Graph each of the following pairs of equations and write the values of the points of intersection:

17. $x^2+y^2=29$, $xy=10$.
18. $3x^2+2y^2=35$, $x^2-2y^2=1$.
19. $x+xy=35$, $y+xy=32$.
20. $x^2+y=7$, $x+y^2=11$.

Note. The student will observe at this point that he can recognize certain types of curves from the appearance of the equations, for example,

(1) The curve $x^2+y^2=r^2$ is a circle whose center is at the origin and whose radius is r.

(2) The curve $(x-a)^2+(y-b)^2=r^2$ is a circle whose center is at the point $(x=a, y=b)$ and whose radius is r.

(3) The curve $x^2-y^2=a$ is a hyperbola which cuts the X axis but not the Y axis; likewise the curve $y^2-x^2=a$ cuts the Y axis but not the X axis.

(4) The curve $xy=\pm a$ is a hyperbola which does not touch either axis.

(5) The curve $ax^2+by^2=c$ is an ellipse.

(6) The curve $y=ax^2+bx+c$ is a parabola.

205. Solution of quadratic equations of one unknown by graphing. We cannot graph equations of one unknown but we can solve equations of the type $ax^2+bx+c=0$ by graphing the equation $y=ax^2+bx+c$. For example, to solve $x^2-2x-15=0$, we graph $y=x^2-2x-15$. Now, we notice that when

$y=0$, the values of x will be the roots of $x^2-2x-15=0$. This method is of use only when the roots are real. If we attempt to solve $x^2+3x+7=0$ by graphing $y=x^2+3x+7$, we will get a parabola that does not cut the X axis, which means, of course, that when $y=0$, there are no real values of x.

Exercise 197

Solve the following quadratic equations by graphing:

1. $x^2-x-12=0$.
2. $x^2-3x-40=0$.
3. $2x^2-7x+3=0$.
4. $2x^2-7x-15=0$.
5. $x^2-9=0$.
6. $x^2=16$.
7. $x^2=20$.
8. $3x^2-7x=6$.
9. $2x^2+3x=20$.
10. $6x^2+7x=5$.

206. Solution of systems of simultaneous quadratic equations.

Not all sets of simultaneous quadratic equations that involve the same unknowns can be solved by the ordinary processes of algebra. However, there are a number of methods, each of which will solve exercises of a particular type. Only the more important types are presented in this chapter.

Case I. A pair of equations, one linear and one quadratic. All of this type can be solved by the method of substitution.

Illustrative example.

(1) $3x+2y=7$.
(2) $2x^2+xy=4$.

From (1) $x=\dfrac{7-2y}{3}$.

Substituting in (2), $\dfrac{2(49-28y+4y^2)}{9}+\dfrac{y(7-2y)}{3}=4$.

Simplifying, $2y^2-35y+62=0$.

Whence $y=2$ or $y=15\frac{1}{2}$.

Substituting in (1) when $y=2$, $x=1$; when $y=15\frac{1}{2}$, $x=-8$.

Check both sets in each equation.

Case II. When all terms of each equation that contain the unknowns are of the second degree. There are several methods that are satisfactory for the solution of exercises of this type. The method of eliminating the constant terms is the only one that will be introduced into the work of this chapter.

Illustrative example.

(1) $2a^2-ab=12.$

(2) $a^2+ab-3b^2=3.$

Multiply (2) by 4, $4a^2+4ab-12b^2=12.$

Subtract (1) $\underline{2a^2-ab \qquad =12}$

$2a^2+5ab-12b^2=0.$

Factoring $(2a-3b)(a+4b)=0.$

Note. Show that we now have two examples under Case I.

Whence $a=\frac{3b}{2}$, $a=-4b$.

Substituting $a=\frac{3b}{2}$ in (1) $\frac{9b^2}{2}-\frac{3b^2}{2}=12.$

Whence $b=\pm2$. Therefore $a=\frac{3b}{2}=\pm3.$

Substituting $a=-4b$ in (1) $32b^2+4b^2=12.$

Whence $b=\pm\frac{1}{3}\sqrt{3}$. Therefore $a=-4b=\pm\frac{4}{3}\sqrt{3}.$

Therefore when $a=\pm3$, $b=\pm2$; when $a=\pm\frac{4}{3}\sqrt{3}$, $b=\pm\frac{1}{3}\sqrt{3}$.

All four sets of answers should be checked in each equation.

Note. When double answers are paired as above, it is understood that when $a=+3$, $b=+2$, and when $a=-3$, $b=-2$.

Case III. When some combination of the two given equations is possible that will make a new equation with the left member a perfect square.

Illustrative example.

(1) $x^2+y^2=17$

(2) $xy=4$

Adding twice (2) to (1), $x^2+2xy+y^2=25.$

Extracting sq. rt., $x+y=\pm5.$

Subtracting twice (2) from (1), $x^2-2xy+y^2=9.$

Extracting sq. rt., $x-y=\pm3.$

Combine $\underline{x+y=\pm5.}$

Note. There are four different ways of combining the two equations, giving four sets of roots. When $x=\pm4$, $y=\pm1$; when $x=\pm1$, $y=\pm4$. Check completely.

Case IV. When the members of one equation are exactly divisible by the corresponding members of the other equation.

Illustrative example.

(1) $x^3+y^3=35$.

(2) $x^2-xy+y^2=7$.

Dividing (1) by (2), $x+y=5$. (3)

Solving (2) and (3) by the method of Case I, when $x=3$, $y=2$; when $x=2$, $y=3$. Check.

Case V. When one of the equations can be solved for some expression containing the unknowns.

Illustrative example.

(1) $x^2+y^2=25$.

(2) $(x+y)^2-12(x+y)+35=0$.

Solving (2) for $(x+y)$, $x+y=7$, or $x+y=5$.

Solving $x+y=7$ with $x^2+y^2=25$; when $x=4$, $y=3$; when $x=3$ $y=4$.

Solving $x+y=5$ with $x^2+y^2=25$; when $x=5$, $y=0$; when $x=0$, $y=5$. Check all sets.

Case VI. Combining the processes of the preceding cases. Many sets of simultaneous equations can be solved by using two or more of the preceding methods, or by applying some simple process of algebra already learned.

Illustrative examples.

I. (1) $x^2+xy+x=18$.

(2) $y^2+xy+y=12$.

Suggestion. Add (1) and (2) and solve for $(x+y)$.

Ans. When $x=3$, $y=2$; when $x=-\frac{18}{5}$, $y=-\frac{12}{5}$.

II. (1) $xy+x-y=7$.

(2) $xy(x-y)=12$.

Suggestion. Put $xy=a$, $x-y=b$.

Then, $a+b=7$ and $ab=12$. Solve for a and b and equate their values with xy and $x-y$. Ans. When $x=4$, $y=1$: when $x=-1$, $y=-4$.

When $x=\pm\sqrt{7}+2$, $y=\pm\sqrt{7}-2$.

III. (1) $5x^2-3xy=-3$.

(2) $2x^2+5xy=36$.

Suggestion. Multiply (1) by 5, (2) by 3, and eliminate xy by addition. Ans. When $x=\pm\sqrt{3}$, $y=\pm 2\sqrt{3}$. (Two sets.) Check.

Exercise 198

Solve and check the following:

1. $x+y=9$
   $x^2+y^2=41.$

2. $a+b=7$
   $ab=12.$

3. $m^2-n^2=45$
   $m-n=5.$

4. $x+y=3$
   $x^2+y^2=45.$

5. $2x+2y=5$
   $xy=1.$

6. $3x+2y=8$
   $3x^2+2y^2=14.$

7. $(x-3)(y+2)=0$
   $x+2y=1.$

8. $xy+4x-2y=8$
   $x+y=5.$

9. $2x-3y=9$
   $xy=-3.$

10. $2x-3y=6$
    $x^2-3xy=0.$

11. $2x+2y=3$
    $2x^2-5xy+2y^2=0.$

12. $x^2-xy=54$
    $xy-y^2=18.$

13. $x^2+y^2=13$
    $xy=6.$

14. $y^2+15=2xy$
    $x^2+y^2-21=xy.$

15. $5x^2-y^2=1$
    $3y^2=xy+10.$

16. $x^2+xy+2y^2=74$
    $2x^2+2xy+y^2=73.$

17. $(x-y)(x+y)=40$
    $(3y+x)(3x+y)=384.$

18. $m^2+n^2+m+n=18$
    $mn=6.$

19. $a^2b^2+ab=6$
    $a^2+b^2=5.$

20. $(x+y)^2+(x+y)=30$
    $xy=6.$

21. $x^2+y^2=25$
    $xy+x+y=19.$

22. $x^2+4y^2=13$
    $xy+x+2y=-2.$

23. $a^3-b^3=26$
    $a-b=2.$

24. $a^3+8b^3=16$
    $a+2b=4.$

25. $x^3+y^3=91$
    $x+y=7.$

26. $x^4+x^2y^2+y^4=21$
    $x^2+xy+y^2=7.$

27. $x^4+x^2y^2+y^4=481$
    $x^2+xy+y^2=37.$

28. $m^3+n^3=2a^3+6a$
    $m^2-mn+n^2=a^2+3.$

29. $r^3+s^3=35$
    $r^2-rs+s^2=7.$

30. $\frac{1}{x^2}-\frac{1}{y^2}=\frac{5}{36}$
    $\frac{1}{x}+\frac{1}{y}=\frac{5}{6}.$

31. $m^4+m^2n^2+n^4=133$
$m^2+mn+n^2=19.$

32. $a^2+b^2=3+ab$
$a^4+b^4=21-a^2b^2.$

33. $a^2-ab+2b^2=16$
$2a^2+3ab-2b^2=8.$

34. $3x^2+y^2=37$
$y^2=29+2x-2x^2.$

35. $a^2+b^2=3ab-1$
$a-b=ab-7.$

36. $3x^2+2y^2=30$
$5x^2-3y^2=-7.$

37. $2x^2+3xy=36$
$3x^2-2xy=15.$

38. $x^2+2y^2+x+2y=10$
$x^2-y^2+x-y=4.$

39. $x+2xy=-6$
$3y-2xy=2.$

40. $4R^2+r^2+4R+2r=6$
$2Rr=1.$

41. $x^2+xy+y^2-63=0$
$x-y+3=0.$

42. $\frac{6}{x}=\frac{y}{10}$
$x-y=11.$

43. $\frac{x}{y}=2$
$xy=8.$

44. $x^2+y^2=500$
$\frac{x+y}{x-y}=3.$

45. $xy+y^2=4$
$2x^2+3xy=27.$

46. $a^4+a^2b^2+b^4=21$
$a^2+ab+b^2=7.$

47. $\frac{y}{x}+\frac{x}{y}=\frac{5}{2}$
$x^2+y^2=5.$

48. $2a^2-3ab+2b^2=43$
$a^2-ab+b^2=39.$

49. $\frac{1}{a}+\frac{1}{b}=\frac{3}{2}$
$\frac{1}{a^2}+\frac{1}{b^2}=\frac{5}{4}$

50. *Solve for x and y:*
$x^2+xy=ab$
$xy+y^2=a^2-ab.$

Exercise 199. Problems

1. Find two numbers whose sum is 2 and the sum of whose squares is 34.

2. The sum of two numbers multiplied by the greater is 28 and the difference of the two numbers is 1. Find the numbers.

3. The difference of two numbers is 2 and the difference of their cubes is 98. Find the numbers.

4. The area of a certain rectangle is 80 square inches. If the width is increased 3 inches and the length is decreased 3 inches, the area will be increased 24 square inches. Find the dimensions.

5. The sum of two numbers is a and the sum of their squares is b. Find the numbers and check the results.

6. The sum of the squares of two numbers exceeds their product by 39, and the difference of the numbers is 33 less than their product. Find the numbers.

7. The sum of two numbers equals the difference of their squares, and 3 times their product equals 4 times the square of the smaller. Find the numbers.

8. The sum of the squares of two numbers added to their sum is 14, and the difference of the squares of the two numbers added to their difference is 10. Find the numbers.

9. The diagonal of a certain rectangle is 13 inches and the area is 60 square inches. Find the dimensions of the rectangle.

10. If a certain number of two digits is divided by the product of the digits, the quotient is 2. If the number is divided by the sum of the digits, the quotient is 4. Find the number.

11. Separate 8 into two parts such that the sum of the cubes of the parts shall be 152.

12. The sum of the squares of two numbers is 170. If the smaller number was 3 greater and the larger number 1 less, the sum of their squares would be 200. Find the numbers.

13. The hypotenuse of a right triangle is 26 inches and the area is 120 square inches. Find the two sides.

14. Some boys hired a motor boat for a trip for $36. Three of the boys were not able to go on which account it cost each of the others $1 more. Find the number of boys that went on the trip and what each one paid.

15. If the average speed of a railway train was increased 10 miles per hour, it would require 2 hours less time to go 400 miles. Find the rate of the train and the time required to make the trip.

16. The three sides of a triangle are 10, 12, and 14 inches. Find the projections upon the longest side of the other two sides.

Suggestion. Draw the altitude to the longest side and study the two right triangles. Let x represent the numerical length of the altitude and y and $14-y$, the segments of the base.

17. Find the area of the triangle of No. 16.

18. Find the area of a triangle whose sides are 8, 9, and 10 inches.

19. A boy lodges his kite in the top of a tree. He stretches the string to a point 14 feet from the foot of the tree and then to a point 64 feet from the foot of the tree. He finds that it requires 30 feet more string to reach the second point. Find the height of the tree.

20. The difference of the squares of two numbers is a, and the quotient obtained by dividing the sum of the numbers by their difference is b. Find the numbers.

21. A guy wire is attached to the top of a flagpole and to a stake in the ground 14 feet from the base of the pole. It is observed that the wire must be lengthened 10 feet to reach a stake 36 feet from the base of the pole. Find the length of the wire and the height of the pole.

CHAPTER XVIII

RATIO, PROPORTION, AND VARIATION

207. Definitions. Define the following: ratio, antecedent, consequent, proportion, terms of a proportion, means, extremes, mean proportional, third proportional, and fourth proportional.

208. Fundamental laws of proportion.

I. In any proportion the product of the means equals the product of the extremes. That is, if $\frac{a}{b}=\frac{c}{d}$, then $ad=bc$.

II. If the product of two factors equals the product of two others, either two may be made the means and the other two the extremes of a proportion. That is, if $ad=bc$, then $\frac{a}{b}=\frac{c}{d}$.

209. Important principles of proportion.

I. If four quantities are in proportion, they are in proportion by alternation. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

Explain by applying the two fundamental laws.

II. If four quantities are in proportion, they are in proportion by inversion. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$. Explain.

III. If four quantities are in proportion, they are in proportion by composition. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

IV. If four quantities are in proportion, they are in proportion by division. That is, if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

210. Geometrical principles.

I. Corresponding sides or altitudes of similar triangles are in proportion.

II. Corresponding sides or diagonals of similar polygons are in proportion.

III. If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.

IV. If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse.

V. The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides of the angle.

VI. If, from a point without a circle, a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.

VII. The areas of two similar polygons have the same ratio as the squares of any two corresponding sides.

VIII. A line is said to be divided in extreme and mean ratio when the longer part is a mean proportional between the whole line and the shorter part.

Exercise 200

Write the following ratios as fractions and simplify:

1. $32 : 48.$ **2.** $57 : 95.$ **3.** $3\frac{5}{8} : 4\frac{1}{7}.$

4. $9a^2b^3 : 21a^5b^2.$ **5.** $x^2-y^2 : (x-y)^2.$

6. $a^3+b^3 : a^4+a^2b^2+b^4.$ **7.** $\frac{x^4-y^4}{x^2-xy+y^2} : \frac{x^2+y^2}{x^3+y^3}.$

Arrange the following proportions by alternation:

8. $\frac{a}{b}=\frac{c}{d}.$ **9.** $\frac{x-y}{2x}=\frac{3y}{x+y}.$ **10.** $\frac{a^2}{c^2}=\frac{b^2}{d^2}.$

Arrange the following proportions by composition and simplify:

11. $\frac{x-y}{y}=\frac{a-b}{b}$. **12.** $\frac{2x-3}{8}=\frac{3x-5}{7-3x}$. **13.** $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

Find the value of x in each of the following proportions:

14. $\frac{x}{3}=\frac{x+8}{9}$. **15.** $\frac{8}{x}=\frac{x}{32}$. **16.** $\frac{7}{x}=\frac{x}{14}$. **17.** $\frac{x+3}{x-5}=\frac{x+7}{x-11}$.

18. $\frac{2x+3}{3x-2}=\frac{x+6}{2x+1}$. **19.** $\frac{3x+5}{5x-2}=\frac{2x+3}{3x-1}$. **20.** $\frac{3-x}{6+x}=\frac{4-x}{8+x}$.

Find the mean proportional between:

21. 7 and 28. **22.** 9 and 3. **23.** $\frac{1}{2}$ and $\frac{1}{4}$.

24. a^3b and ab^5. **25.** a^2-b^2 and $a+b$.

26. m^3-n^3 and m^4-n^4.

27. Given $xy=mn$, form a proportion from x, y, m, and n.

28. Given $a^2-b^2=x^4+x^2+1$, form a proportion.

29. The ratio of 625 to x^3 is 5. Find x.

30. The mean proportional between 5 and 9.6 is x. Find x.

Exercise 201. Problems

1. What number must be subtracted from 50, 45, 75, and 65 so that the remainders shall form a proportion?

2. The ratio of the rate of a local train to that of an express train is 2 : 3. If the local train runs 30 miles per hour, find the rate of the express train.

3. The ratio of the length of Lake Erie to the length of Lake Michigan is 3 : 4. What is the length of each if Lake Michigan is 90 miles longer than Lake Erie?

4. A base ball player threw a ball 210 feet. The ratio of this distance to the distance usually considered the limit for players is 5 : 7. Find the usual limit for ball players.

5. Two automobile racers start at the same time and travel in the same direction, their rates being in the ratio 2 : 3. In 5 hours they are 100 miles apart. Find the rate of each.

6. Each of 3 camps had the same number of boys and all were fed from food supplied by the first two. If one of them furnished $\frac{7}{8}$ as much as the other, and the third camp paid the others \$30, how was it divided between the first two camps?

7. The sides of a triangle are 5, 6, and 7 inches, respectively, and in a similar triangle the side corresponding to the shortest side of the first is 10 inches. Find the other sides.

8. The sides of a triangle are 2, 3 and 4 inches, respectively. The perimeter of a similar triangle is 108 inches. Find the lengths of the sides of the second triangle.

Suggestion. The ratio of the perimeters of similar triangles is the same as the ratio of a pair of corresponding sides.

9. The shadow of a tree is 100 feet long. If the shadow of a vertical pole 5 feet in length is 4 feet long, what is the height of the tree?

10. The sides of a triangle are 8, 12, and 15 inches. Find the segments of the longest side made by the bisector of the opposite angle.

11. The sides of a triangle are a, b, and c. Find the segments of each side made by the bisectors of the opposite angles.

12. Use the results of No. 11 to find the segments of the sides of the triangle, which are 16, 18, and 20 inches, made by the bisectors of the opposite angles.

13. If a boy weighing 125 pounds at a distance of 5 feet from the fulcrum is to balance a 75 pound boy, where should the second boy be placed?

Note. If weights are placed on the two ends of a beam and the beam turns about a pivot or fulcrum, the beam will balance when $\frac{w_1}{w_2}=\frac{d_2}{d_1}$. That is, the weights are in inverse ratio to the distances.

14. Where should the fulcrum be placed to balance two boys weighing 120 and 90 pounds respectively, if the beam is 16 feet long?

15. A crowbar 5 feet long is used to lift a stone weighing 780 pounds. The fulcrum of the crowbar is 8 inches from the end of the bar. What force must be applied at the end of the bar to lift the stone?

16. The plan for a building is drawn to a scale of $\frac{1}{4}$ inch to 1 foot. The length of a certain beam is shown on the plan to be $3\frac{1}{4}$ inches. Find the length of the beam.

17. If in a map the distance between two cities 540 miles apart is $2\frac{1}{4}$ inches, what is the distance between two cities which are $3\frac{1}{2}$ inches apart on the map?

18. The perimeter of a triangle is 63 inches and the ratio between the longest and shortest side is $3 : 2$, while the third side is a mean proportional between the other two. Find the sides of the triangle.

19. Wishing to determine the height of a flag staff, a man noticed that by holding a 12 inch ruler vertically in front of his line of sight, at a distance of 2 feet from his eye, the ruler and the flag staff subtended the same angle. How tall was the staff if the man was 200 feet from the base of the staff?

20. The corresponding sides of two similar triangles are 5 inches and 8 inches. If the area of the first is 50 square inches, what is the area of the second?

21. The area of a square is 144 square inches. What is the ratio of a side of this square to the side of another square whose area is 9 times as great?

22. The areas of two similar triangles are 50 and 200 square inches, respectively. If the base of the first is 10 inches, what is the base of the second?

23. The altitude to the hypotenuse of a right triangle is 9 inches. If the entire length of the hypotenuse is 30 inches, find the segments of the hypotenuse.

24. A secant to a circle is 63 inches long. If a tangent to the circle from the same point is 21 inches, find the length of the external part of the secant.

25. A line 8 inches long is divided in extreme and mean ratio. Find the parts of the line.

26. A diameter of a circle is 20 inches in length. A perpendicular is drawn to the diameter from a point on the circle. If the length of the perpendicular is 8 inches, find the segments of the diameter.

Note. The perpendicular to the diameter from a point on the circle is a mean proportional between the segments of the diameter.

27. Two sides of a triangle are 12 and 18 inches, respectively. A line parallel to the base divides the shorter side in the ratio 1 : 2. Find the segments of the longer side.

28. Two corresponding sides of two similar polygons are 5 and 7 inches, respectively. If the area of the smaller polygon is 150 square inches, find the area of the larger.

29. A father divides $3000 among his three sons so that the eldest son receives 20% more than the second, and the second receives 25% more than the youngest. Find the amount each one receives.

30. The difference of the squares of two numbers in the ratio 5 : 2 is 168. Find the numbers.

211. Variation. When is one variable said to be a function of another? (See § 157.)

In the equation $y=x^2$, y is a function of x. By solving the equation for x in terms of y, we get $x=\pm\sqrt{y}$, which expresses x as a function of y.

One variable is said to **vary as** another when the first is equal to the product of the second and a **constant.**

In the equation $x=ky$, x varies as y since x equals the product of y and a constant.

One variable is said to **vary inversely** as another when the first is the quotient of a constant and the second variable.

In the equation $x=\frac{k}{y}$, x varies inversely as y. Evidently, this may also be written $xy=k$.

One variable is said to **vary jointly** as two others when the first variable equals the product of the others and a constant.

In the equation $x = kyz$, x varies jointly as y and z.

212. Many of the formulas of mathematics may be expressed in terms of variation.

For example, $A = \pi r^2$ expresses the fact that the area of a circle varies as the square of the radius.

Similarly, $V = \frac{4}{3}\pi r^3$ expresses the fact that the volume of a sphere varies as the cube of the radius.

Also, $A = \frac{1}{2}bh$ expresses the fact that the area of a triangle varies jointly as its base and altitude.

It is customary to represent the **constant** in each problem of variation by the letter k.

Exercise 202

1. If x varies as y, and $x = 14$ when $y = 7$, find x when $y = 9$.

Solution. Substituting $x = 14$ and $y = 7$ in $x = ky$, gives $14 = 7k$, or $k = 2$. The reduced equation is $x = 2y$, and when $y = 9$, $x = 18$.

2. If x varies as y and $x = \frac{1}{2}$ when $y = \frac{2}{3}$, find x when $y = 5\frac{2}{3}$.

3. In the formula for the volume of the sphere, what is the constant and what are the variables? Find the constant correct to .0001.

4. If x varies inversely as y, and if $x = 8$ when $y = 3$, find x when $y = \frac{4}{3}$.

5. Write a formula to show that the pressure in pounds of the wind on a sail varies jointly as the area of the sail and the square of the wind's velocity.

6. The distance that a body falls from a state of rest varies as the square of the time it falls. Express this fact by an equation and determine the constant, k, if the body falls the first foot in $\frac{1}{4}$ of a second.

7. The intensity of light varies inversely as the square of the distance of the light from the surface illuminated. If a 20-candle power light 4 feet from the page furnishes a comfortable reading light, how far from the page must a 40-candle power light be placed? A 100-candle power?

8. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 125 pounds on the surface of the earth (assume it to be 4000 miles from the center). What will be its weight 1000 miles above the surface? 4000 miles?

9. How far above the surface of the earth must the object in No. 8 be carried to reduce its weight one-half?

10. The safe load, w, of a horizontal beam supported at each end is expressed by the formula $w=\frac{kbd^2}{l}$, where b is the breadth, d the depth, and l the length, or distance between supports. If a 2 by 6 yellow pine joist set horizontally on its edge on supports that are 10 feet apart carries safely a load of 600 pounds, what would be the safe load if its supports were 12 feet apart?

11. Would the joist in No. 10, if placed on its side across a stream 12 feet wide, be safe for a man weighing 150 pounds?

12. What is the safe load for a 2 by 10 joist of the same material as in No. 10 placed on its edge and resting on supports 16 feet apart? Of a 3 by 12 resting on supports 20 feet apart?

13. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once per second, find the length of a pendulum that vibrates once in 2 seconds. Once in 4 seconds. Once in $\frac{1}{2}$ a second.

14. What is the time of vibration of a pendulum 50 centimeters long? Of a pendulum 81 centimeters long? Of a pendulum 1000 centimeters long?

CHAPTER XIX

PROGRESSIONS

213. Definitions. A **series** is a succession of related terms whose values are determined according to some law.

A series is **finite** or **infinite** according as the number of its terms is finite or infinite.

The variety of different series is unlimited, as the succeeding paragraphs will show.

Exercise 203

Write three or four additional terms for each of the following series:

1. 1, 2, 3, ...

2. 1, 3, 5, 7, ...

3. 8, 11, 14, 17, ...

4. 2, 4, 8, 16, ...

5. $1+x+x^2+x^3$, ...

6. $1-x^2+x^4-x^6$, ...

7. 1, 4, 16, ...

8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, ...

9. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, ...

10. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, ...

11. $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}$, ...

214. The expression $\sum_{n=1}^{n=5}(n^2+1)$ is taken to mean the sum of the series of five terms obtained by substituting in (n^2+1) the numbers from 1 to 5 inclusive. This is a convenient way for writing in compact form any series. Written as an identity we have, $\sum_{n=1}^{n=5}(n^2+1) \equiv 2+5+10+17+26.$

The expression $n=\infty$ means that n **becomes infinite,** or that its value increases beyond all bounds.

Therefore $\sum\limits_{n=1}^{n=\infty}$(any expression containing n) would represent an infinite series, or one the number of whose terms is unlimited.

$$\text{For example, } \sum_{n=1}^{n=\infty}\frac{n}{n^2+1}=\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\frac{4}{17}\cdots\cdots\cdots\cdots$$

Exercise 204

Expand the following series:

1. $\sum\limits_{n=1}^{n=5} n(n+1)$. Ans. $\sum\limits_{n=1}^{n=5} n(n+1)=2+6+12+20+30.$

2. $\sum\limits_{n=1}^{n=5}(n+2)$. **3.** $\sum\limits_{n=1}^{n=4}(5n-2)$. **4.** $\sum\limits_{n=1}^{n=6}\frac{(n+1)(n+2)}{n}$.

5. $\sum\limits_{n=1}^{n=5}\frac{n(n-3)}{n+1}$. **6.** $\sum\limits_{n=1}^{n=6}\frac{n(n+1)^2}{2}$. **7.** $\sum\limits_{n=1}^{n=6}\frac{n(2n+3)}{n+1}$.

Write the first five terms of each of the following series:

8. $\sum\limits_{n=1}^{n=\infty}\frac{1}{n+2}$. Ans. $\sum\limits_{n=1}^{n=\infty}\frac{1}{n+2}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\cdots\cdots$

9. $\sum\limits_{n=1}^{n=\infty}\frac{n}{2n+3}$. **10.** $\sum\limits_{n=1}^{n=\infty}\frac{1}{n^2+n}$. **11.** $\sum\limits_{n=1}^{n=\infty}\frac{1}{2^n}$.

12. $\sum_{n=1}^{n=\infty} (\frac{2}{3})^n$. 13. $\sum_{n=1}^{n=\infty} (n+1)^n$. 14. $\sum_{n=1}^{n=\infty} \frac{n}{(n+1)^n}$.

215. Arithmetic progression. If each term after the first of a series is found by adding a **constant difference** to the preceding term, the series is called an **arithmetic progression.**

The **first term** of an arithmetic progression is represented by a; the **common difference,** by d; the **last term,** by l; and the number of terms, by n. The terms in order are represented by $a, a+d, a+2d, a+3d, \ldots$

What is the 5th term? The 12th term? The 20th term?

From the preceding we obtain the formula,

$l=a+(n-1)d$. (I)

Solve this formula for a, n, and d.

We now have at our disposal four formulas which will enable us to determine any one of the four literal numbers, a, l, n, or d, when the other three are given.

Exercise 205

1. Given $a=3$, $d=4$, $n=8$; find l.
2. Given $d=2$, $a=13$, $l=27$; find n.
3. Given $l=31$, $d=4$, $n=6$; find a.
4. Given $l=48$, $a=8$, $n=9$; find d.
5. Given $a=-3$, $d=2\frac{1}{2}$, $n=10$; find l.
6. Given $a=4$, $d=-5$, $l=-26$; find n.
7. Given $l=14\frac{1}{3}$, $n=9$, $d=2$; find a.
8. Given $a=-5$, $l=-26$, $n=8$; find d.
9. A ball rolling down an inclined plane goes 1 foot the first second, 3 feet the second second, and 5 feet the third second. Find how many feet it will go the eighth second.
10. A body falling freely falls 16 feet the first second, 48 the next, and 80 the next. Find how many feet it will fall the tenth second.

11. There are four numbers in arithmetic progression whose sum is 38. The product of the second and third exceeds the product of the first and fourth by 18. Find the progression.

Suggestion. Let $a-3d$, $a-d$, $a+d$, and $a+3d$ represent the numbers.

12. The sum of the first three terms of an arithmetic progression is 15 and the fifth term exceeds the second by 21. Find the eighth term.

216. Arithmetic means. The first and last terms of a series are called the **extremes** and the remaining terms are called the **means.** The problem of inserting a number of arithmetic means between two given extremes resolves itself into finding d, when a, l, and n are given, and writing the series.

Exercise 206

1. Insert 4 arithmetic means between 3 and 23. Solving, $d=4$. Ans. 3, 7, 11, 15, 19, 23.

Suggestion. $n=6$.

2. Insert 5 arithmetic means between 7 and 43.

3. Insert 4 arithmetic means between -7 and 23.

4. Insert 7 arithmetic means between 2 and 4.

5. Insert 3 arithmetic means between a and b and use the results as formulas to insert 3 arithmetic means between 5 and 21.

6. Find the arithmetic mean between a and b; between 5 and 37; between $3x$ and $2y$.

7. A board is held in place by two nails 15 inches apart. Show how to place 8 more nails between the two at equal intervals.

8. The arithmetic mean of two numbers is 12 and the sum of their squares is 306. Find the numbers.

217. The sum of an arithmetic series. If we represent the last term of an arithmetic series by l, then $l-d$ represents the term before the last, $l-2d$ the term preceding that, etc. Then the sum of any series can be expressed as follows:

$$s=a+(a+d)+(a+2d). \quad . \quad . \quad . \quad +(l-2d)+(l-d)+l. \qquad (1)$$

Writing the series in reverse order,

$$s=l+(l-d)+(l-2d). \quad . \quad . \quad . \quad +(a+2d)+(a+d)+a. \qquad (2)$$

adding (1) and (2),

$$2s=(a+l)+(a+l)+(a+l). \quad . \quad . \quad . \quad +(a+l)+(a+l)+(a+l),$$

or $2s=n(a+l)$.

Therefore $s=\frac{n}{2}(a+l)$. (II)

If we substitute the value of l in the formula $l=a+(n-1)d$ for l in II, we get $s=\frac{n}{2}[2a+(n-1)d]$. (III)

If any three of the five numbers, a, l, d, n, and s of an arithmetic progression are given, the remaining two can be found by means of the preceding formulas, I, II, and III.

Exercise 207

Use formulas I, II, and III to solve the following:

1. Given $a=2$, $l=23$, $n=8$; find d and s.
2. Given $a=-5$, $l=27$, $d=4$; find n and s.
3. Given $a=8$, $l=-10$, $s=-7$; find n and d.
4. Given $a=-9$, $n=6$, $d=7$; find l and s.
5. Given $a=2\frac{1}{2}$, $n=10$, $s=137\frac{1}{2}$; find l and d.
6. Given $a=5$, $d=3$, $s=75$; find l and n.

Suggestion. Substitute first in III and solve the resulting quadratic for n. Will both roots satisfy the problem?

7. Given $l=-26$, $n=8$, $d=-3$; find a and s.
8. Given $l=6$, $n=9$, $s=30$; find a and d.
9. Given $l=14$, $d=4$, $s=24$; find a and n.
10. Given $n=9$, $d=2$, $s=57$; find a and l.

11. A ball rolling down an inclined plane rolls 3 feet the first second, 9 feet the next and 15 the next. Find how far it will roll in 8 seconds.

12. A body falling freely falls 16 feet the first second, 48 feet the next, and 80 the next. Find how far it will fall in 5 seconds; in 8 seconds; in t seconds.

218. Geometric progressions. A **geometric progression** is a series such that the quotient of any term divided by the preceding term is a constant number. This **constant ratio** is represented by r. The letters a, l, n, and s are used in geometric progressions with the same meaning as in arithmetic progressions. 5, 10, 20, 40, . . . is a geometric progression.

If a, ar, ar^2, ar^3 . . . represent the terms of a geometric progression, what is the 8th term? The 10th term? The nth term? The last question suggests the formula, $l = ar^{n-1}$. (I)

Solve this formula for a; for r. Ans. $r = \sqrt[n-1]{\frac{l}{a}}$.

As in arithmetic progressions, the first and last terms are called the **extremes** and the remaining terms the **means.**

Exercise 208

1. Find the 8th term of the series 3, 6, 12 . . . .

2. Find the 5th term of 5, -15, 45 . . . .

3. Given $a = -2$, $r = 4$, $n = 5$; find l.

4. Given $l = 36$, $n = 4$, $r = 3$; find a.

5. Given $l = 56$, $a = 7$, $n = 4$; find r.

6. Insert 3 geometric means between 3 and 48.

Suggestion. Show $r = \pm 2$, and write both series.

7. Show that the logarithms of the terms of a geometric series form an arithmetic series.

8. Show that $n = \frac{\log l - \log a}{\log r} + 1$.

9. The first two terms of a geometric series are x and y. Find the third term.

10. Find the 18th term of 2, 6, 18, ...

Suggestion. Use logarithms.

11. Find the 20th term of 27, 18, 12, 8, ...

12. If one dollar is placed at compound interest at 6%, it is worth, at the end of the first year, \$1.06; at the end of the second year, $\$(1.06)^2$; and at the end of the third year, $\$(1.06)^3$. Explain. Find by logarithms its value at the end of the 12th year.

13. Find the value of one dollar at compound interest at 5% at the end of 100 years.

14. Find the value of \$100 at compound interest at 6% at the end of 8 years.

219. Sum of a geometric series. The formula for the sum of a geometric series is discovered by the following process;

(1) $s=a+ar+ar^2+ar^3 \quad . \ . \ . \quad ar^{n-1}$.

Multiplying both members of (1) by r,

(2) $rs=ar+ar^2+ar^3 \quad . \ . \ . \quad ar^{n-1}+ar^n$.

Subtracting (1) from (2), $rs-s=ar^n-a$.

Whence $s=\frac{ar^n-a}{r-1}$. (II)

Since $l=ar^{n-1}$, $rl=ar^n$; then by substitution we have, $s=\frac{rl-a}{r-1}$. (III)

Exercise 209

1. Solve Formula II for a.

2. Solve Formula III for a; for r; for l.

3. Find the sum of eight terms of the series 1, 2, 4, 8, . . .

4. Given $a=3$, $r=3$, $n=5$. Find s.

5. Find the sum of 6 terms of 3, $3\sqrt{3}$, 9, ...

6. Find the sum of 7 terms of 2, -4, 8, -16, ...

7. Find the sum of 5 terms of 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, ...

8. The fourth term of a geometric progression is 12 and the sixth term is 27. Find the sum of the first 6 terms.

9. Find the sum of seven terms of $1+x+x^2+x^3$ . . . . Reduce the result to simplest form by dividing the numerator by the denominator.

10. Find the sum of five terms of $1-x+x^2$ . . . . Reduce the result to simplest form.

11. A boy asking for a position is offered one cent for his first day's work, and for each succeeding day double that of the previous day. He works 12 days. Find the total amount he receives.

12. A person writes the same letter to each of three of his friends, and asks them each one to write the same letter to three friends, and so on till the tenth set of letters has been written. Find the total number of letters that will be written in the entire chain.

220. Infinite geometric series. When in any given series r is less than one, it will be observed that each term is less than the preceding term, and as the number of terms becomes greater, the last term becomes smaller and smaller and approaches zero as a limit. In the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, it is very easy to find a term that is less than any given amount, as, for example, .001. See § 236.

Find the sum of six terms of this series; eight terms; twelve terms. What do you notice about the sum?

Find by logarithms the 40th term of the series 10, 9, 8.1, ... Find the 100th term.

In the formula $s=\frac{ar^n-a}{r-1}$, when r is less than one and the number of terms is unlimited, it will be observed that ar^n approaches zero as a limit. Therefore for any decreasing infinite series $s \doteq \frac{-a}{r-1}$, or $s \doteq \frac{a}{1-r}$. (Read $\doteq$ "approaches as a limit.")

221. Repeating decimals. A common fraction in its lowest terms, whose denominator contains prime factors other than 2 and 5, cannot be expressed exactly in decimal form. When we attempt to reduce such a fraction to decimals, we find that the same groups of digits occur repeatedly. As, for example,

$\frac{1}{3} = .3333 \ldots$

$\frac{5}{11} = .454545 \ldots$

$\frac{3}{7} = .428571428571 \ldots$

Any number in repeating decimal form may be considered as an infinite geometric series. As, for example,

$$.3333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$

Here $a = \frac{3}{10}$, $r = \frac{1}{10}$. Then $s = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$.

Exercise 210

Find the sum of each of the following infinite series:

1. $2, 1, \frac{1}{2}, \ldots$ Ans. 4.
2. $3, 2, \frac{4}{3}, \ldots$
3. $3, -2, \frac{4}{3}, -\frac{8}{9}, \ldots$
4. $27, 18, 12, \ldots$
5. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$
6. $6, 3\sqrt{2}, 3, \ldots$
7. $4, \dfrac{2}{\sqrt{3}-1}, \dfrac{1}{4-2\sqrt{3}}$.
8. $1, x, x^2, \ldots$ (When x is less than 1.)

Find the values of the following repeating decimals:

9. $.4545 \ldots$

Suggestion. As a series $.4545 \ldots = \frac{45}{100} + \frac{45}{10000} \ldots$ Ans. $\frac{5}{11}$.

10. $.6363 \ldots$
11. $3.16363 \ldots$

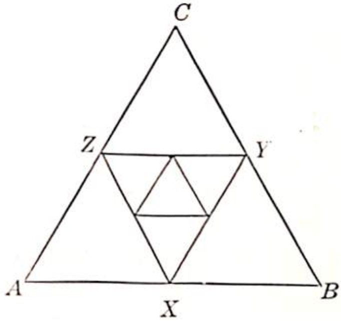
Note. This is 3.1 plus the series $\frac{63}{1000} + \frac{63}{100000} \ldots$ Ans. $3\frac{9}{55}$.

12. 8.1666 . . . . **13.** 5.0333 . . . .

14. .243243 . . . . **15.** 1.44144144 . . . .

16. If a ball, dropped from a height of 60 feet, rebounds 30 feet on striking the ground, and again rebounds 15 feet, and so on, how far will it travel before coming to rest?

17. One side of the equilateral triangle ABC is 10 inches. The mid-points of the sides are connected forming the triangle XYZ, and so on. If this process is continued indefinitely, find the total length of the lines.



Exercise 211. Review

1. Suppose that every term of an arithmetic progression is multiplied by k; is the result an arithmetic progression?

2. Show that the quotients form a geometric progression when each term of a geometric progression is divided by the same number.

3. What is the sum of the first 200 numbers that are divisible by 5?

4. How many multiples of 7 are there between 350 and 1210?

5. The sum of four numbers of an arithmetic progression is 0, and the sum of their squares is 125. Find the numbers.

6. Show that the sum of $2n+1$ consecutive integers is divisible by $2n+1$.

7. Show that the sum of the arithmetic progression 1, 3, 5, 7, . . . is n^2 where n is the number of terms.

8. Find the sum of all numbers under 200 that are divisible by 3 and not divisible by 2.

9. The product of the 3 terms of a geometric progression is 512. If the first term is 1, find the second.

10. The sum of the first eight terms of a geometric progression is 17 times the sum of the first four. Find r.

11. Find the sum of the first six terms of a geometric progression if the second term is 5 and the fifth, 625.

12. Three numbers form a geometric progression and they are the second, fourth, and ninth terms of an arithmetic progression whose first term is 1. Find d.

13. What is the fourth term of a geometric progression if the second is $\frac{1}{5}$ and the fifth is 625?

14. Find the sum of five consecutive powers of 3 beginning with the first.

15. Of three numbers in geometric progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. Find the numbers.

16. Three numbers form a geometric progression. If 2 is subtracted from the first, 4 from the second, and 13 from the third, the results form an arithmetic progression, the sum of whose terms is 30. Find the arithmetic progression.

17. To what sum will \$1 amount at 4% compound interest in 5 years?

18. The number of oranges in a pile in the form of a triangular pyramid is $1+(1+2)+(1+2+3)+\ .\ .\ .$ depending on the number of layers. How many oranges in a pile of 10 layers?

19. If a, b, and c form a geometric progression, show that $\frac{1}{b-a}$, $\frac{1}{2b}$, and $\frac{1}{b-c}$ form an arithmetic progression.

20. A farmer hires a laborer for the summer at a beginning salary of \$50 a month, with either a raise of \$10 per month, after the first month, or a raise of \$2.50 every two weeks after the first half month. If 4 weeks are considered a month, which is the better proposition for the laborer, provided he will work 20 weeks?

21. A man travels at a constantly increasing rate. He goes 1 mile the first hour, 3 miles the next, 5 the next, and so on. How far will he go in $6\frac{1}{4}$ hours?

22. A well-drilling company in estimating the number of linear feet of casing in stock, find they have 10 piles of 30-foot sections each pile being arranged in triangular form, 12 pipes in the first layer, 11 in the next, etc. How many feet of casing in their stock?

23. In a potato race, the potatoes are placed in a row, the first 10 feet from a box and the remaining ones at intervals of 4 feet. There are 25 potatoes in all. The contestant starts at the box and fetches them one at a time. How far has he run when all the potatoes are placed in the box?

24. Neglecting the resistance of the air a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 feet the third, etc. How far will it fall in 6 seconds? in 12 seconds?

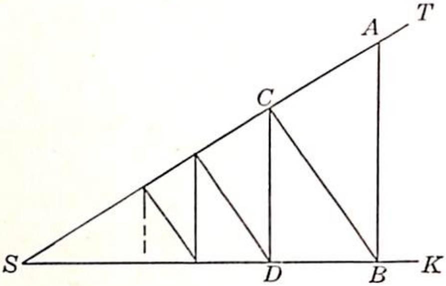
25. The increase in velocity as represented in No. 24 is called acceleration due to gravity and is represented by the literal number g. Using this value for d, how far does the body fall in t seconds? The result is a well-known formula in Physics. Ans. $s=\frac{1}{2}gt^2$.

26. If a falling body is given an initial velocity of v feet per second, then the distance it falls in t seconds is vt feet plus the distance it would fall if starting from rest, i. e., $s=vt+\frac{1}{2}gt^2$.

How long will it take a body to fall 3918 feet, if given an initial velocity of 20 feet per second?

27. What must be the initial velocity of a falling body in order that it shall fall 1200 feet in $7\frac{1}{2}$ seconds?

28. A pulley rolling on a cable which is inclined at an angle of 30°, goes down at the rate of 7.25 feet the first second and in each succeeding second 14.5 feet more than in the preceding one. How long will it take the pulley to reach the end of a 261-foot cable?



29. In the accompanying figure, AB is perpendicular to SK, BC is perpendicular to ST, CD to SK, ... If AB is 12 inches and CB is 10 inches, what is the sum of all the perpendiculars?

30. If an air pump draws out at each stroke 1/10 the volume of the air in the bell jar, what fractional part of the air will remain in the jar at the end of the tenth stroke?

31. There is an Eastern legend that the ruler for whom the game of chess was invented foolishly agreed to pay the inventor 1 grain of wheat for the first square on the board, 2 for the second, 4 for the third, 8 for the fourth, . . .

Determine by logarithms the number of digits in the number of grains of wheat that the inventor should have received and the four figures at the extreme left of this number.

CHAPTER XX

THE BINOMIAL THEOREM

222. The powers of $(a+b)$ and of $(a-b)$.

The following powers are obtained by multiplication:

$(a+b)^1=a+b.$

$(a+b)^2=a^2+2ab+b^2.$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

Similarly, find the corresponding powers of $a-b$.

In the powers of $(a\pm b)^n$, when n is a positive integral number, observe the following:

(1) The number of terms is one greater than n.

(2) The first term is a^n and the exponent of a decreases by unity in each succeeding term.

(3) The exponent of b is unity in the second term and increases by unity in each succeeding term.

(4) The coefficient of the second term is n, and if the coefficient of any term is multiplied by the exponent of a in that term, and the product divided by the number of the term, the quotient is the coefficient of the next term.

(5) The signs of $(a+b)^n$ are all $+$ and the signs of $(a-b)^n$ are alternately $+$ and $-$.

Illustrative examples.

1. Expand $(a+b)^7$.

Solution. The first term is a^7 and the second term is $7a^6b$. The coefficient of the third term is found by multiplying 7 by 6 and dividing the product by 2. The complete third term is $21a^5b^2$.

Continue in the same way to find succeeding terms. The expansion is,

$$(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

2. Expand $(2x^2-3y^3)^3$.

Solution. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

Using this as a formula, we get,

$$(2x^2-3y^3)^3=(2x^2)^3-3(2x^2)^2(3y^3)+3(2x^2)(3y^3)^2-(3y^3)^3=$$
$$8x^6-36x^4y^3+54x^2y^6-27y^9.$$

Exercise 212

Expand the following:

1. $(a-b)^6$. 2. $(x+y)^7$. 3. $(x-y)^8$. 4. $(2x+3y)^4$.
5. $(3a-2b)^5$. 6. $(2n^2+3m^2)^5$. 7. $(3x^2-5y)^4$.
8. $(7R^2-6r)^3$. 9. $(3m^2n-4)^5$. 10. $(6xy^3-2a^2b)^4$.
11. $\left(2x+\frac{y}{2}\right)^4$. 12. $\left(\frac{x}{y}+\frac{m}{n}\right)^3$. 13. $\left(\frac{x^2}{y^2}-\frac{y^2}{x^2}\right)^4$.

223. Any required term. If the coefficients of the terms of the binomial expansion are determined by (4) of Section 222, and the coefficients are left in the fractional form, a simple process for writing any term is discovered. For example,

$$(a\pm b)^6=a^6\pm\frac{6}{1}a^5b+\frac{6\cdot5}{1\cdot2}a^4b^2\pm\frac{6\cdot5\cdot4}{1\cdot2\cdot3}a^3b^3+$$
$$\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}a^2b^4\pm\frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot3\cdot4\cdot5}ab^5+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\cdot5\cdot6}b^6.$$

Similarly, write the expansion of $(a\pm b)^7$.

Observations.

(1) The factors of each numerator are $n\,(n-1)\,(n-2)\ldots$

(2) The factors of each denominator are $1\cdot2\cdot3\ldots$

(3) The number of factors in each numerator and denominator is one less than the number of the term.

(4) The exponent of b is one less than the number of the term and the exponent of a is found by subtracting the exponent of b from n.

(5) The only signs that are negative are the signs of the even numbered terms of $(a-b)^n$.

Note. These observations may all be condensed into the **formula:** The rth term of $(a\pm b)^n\equiv$

$$\frac{n(n-1)\,(n-2)\ldots\text{ to }(r-1)\text{ factors}}{1\cdot2\cdot3\ldots\text{ to }(r-1)\text{ factors}}a^{n-r+1}b^{r-1}.$$

Illustrative examples.

1. Find the 5th term of $(x+y)^8$.

Solution. 5th term of $(x+y)^8=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}x^4y^4=70x^4y^4$.

2. Find the 6th term of $(2n-3m)^9$.

Solution. 6th term of $(2n-3m)^9=-\frac{9\cdot8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4\cdot5}(2n)^4(3m)^5$
$=-489888n^4m^5$. (Explain the negative sign.)

Exercise 213

1. Find the 7th term of $(a+b)^{11}$.
2. Find the 8th term of $(x-y)^{12}$.
3. Find the 5th term of $(2a-b)^9$.
4. Find the 4th term of $(3n+2m)^7$.
5. Find the 3rd term of $(2a-\sqrt{b})^6$.
6. Find the 6th term of $\left(3a^2-\frac{b}{2}\right)^7$.
7. Find the 4th term of $\left(\frac{x}{2}-\frac{y}{3}\right)^6$.

It is explained in the work of the preceding paragraphs that the following formula holds true for the expansion of $(a\pm b)^n$, when n is a positive integral number.

$$(a\pm b)^n\equiv a^n\pm\frac{n}{1}a^{n-1}b+\frac{n(n-1)}{1\cdot2}a^{n-2}b^2\pm\frac{n(n-1)(n-2)}{1\cdot2\cdot3}a^{n-3}b^3\ \ldots.$$

It will be noticed that the number of terms is one more than n and, if one should attempt to find more than $n+1$ terms by the formula, as the 5th term of $(a+b)^3$, one of the factors of the numerator of the coefficient is zero which makes the term zero.

The series of terms in the right member of the above is called the expansion of $(a\pm b)^n$. The series is finite and contains $n+1$ terms only when n is a positive whole number. The whole identity is called the **binomial theorem.**

224. Binomial theorem, exponent fractional or negative. It is proved in higher mathematics that the binomial theorem is true for fractional and negative exponents when the first term of the binomial is arithmetically greater than the second. The expansion of any binomial with a fractional or negative exponent gives an infinite series, which can be interpreted if the larger term, arithmetically, of the binomial is the first term.

Illustrative examples.

1. Expand $(1-x)^{-1}$.

Solution. Substituting in the formula,

$$(1-x)^{-1}=1-(-1)x+\frac{(-1)(-2)}{1\cdot 2}x^2-\frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}x^3+\ .\ .\ .$$

$$=1+x+x^2+x^3+\ .\ .\ .\ .$$

Can you write additional terms by inspection?

It will be noticed that the expansion of $(1-x)^{-1}$ is an infinite geometrical progression with the first term 1 and the ratio x. This series is of meaning to us only when x is less than 1. Applying the formula for finding the sum of the geometric series $1+x+x^2+x^3\ .\ .\ .$ we get $\frac{1}{1-x}$ which is another way of writing $(1-x)^{-1}$.

2. Expand $(1+x)^{\frac{1}{2}}$.

Solution. $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}x^3\ \ .\ .\ .$

$$=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\ \ .\ .\ .$$

3. Write the rth term of $(1+x)^{\frac{1}{2}}$.

Solution. By a careful study of Example 2, it will be observed that the factors of the numerator of the rth term of $(1+x)^{\frac{1}{2}}$ are $(\frac{1}{2})\ (-\frac{1}{2})\ (-\frac{3}{2})\ (-\frac{5}{2})\ .\ .\ .$ to $r-1$ factors, while the factors of the denominator are $1\cdot 2\cdot 3\ .\ .\ .$ to $r-1$ factors. If each factor of the numerator and denominator is multiplied by 2, the rth term reduces to

$$\frac{(1)(-1)(-3)(-5)\ .\ .\ .\ r-1 \text{ factors}}{2\cdot 4\cdot 6\cdot 8\ .\ .\ .\ r-1 \text{ factors}}\,x^{r-1}.$$

Exercise 214

1. Expand $(a-b)^8$ to 4 terms.

2. Expand to 4 terms $(1+x)^{-1}$. Check by performing the division $\dfrac{1}{1+x}$.

3. Expand to 4 terms $(1-x)^{-2}$. Check by performing the division $\dfrac{1}{1-2x+x^2}$.

4. Expand to 4 terms $(1+x)^{\frac{2}{3}}$.

5. Expand to 4 terms $(1-2x)^{\frac{3}{4}}$.

6. Expand to 4 terms $(8x^3-2y)^{\frac{4}{3}}$.

7. Expand to 3 terms $(1+x)^{-\frac{1}{2}}$.

8. Expand to 4 terms $(x-2y)^{-3}$.

9. Expand to 4 terms $(4x^2-5)^{\frac{1}{4}}$.

10. Expand to 4 terms $(2x+y)^{10}$.

11. Expand to 4 terms $(5x^2+3b)^5$.

12. Find the 6th term of $(x-2y)^{10}$.

13. Find the 7th term of $(3x-2y)^{11}$.

14. Find the 6th term of $(1-2x)^{-1}$.

15. Find the 5th term of $(1+x)^{\frac{1}{2}}$.

16. Find the 8th term of $(2-x)^{-\frac{1}{2}}$.

17. Expand to 3 terms $(1+2x)^{\frac{1}{2}}$ and check by finding the square root of $1+2x$.

225. Square root and cube root. The binomial theorem and its applications furnish us with the solution of many practical problems. One of these is the extraction of any root of any number. Square root and cube root only will be treated here. As a preliminary step, we expand the following:

$$(1\pm x)^{\frac{1}{2}}=1\pm\tfrac{1}{2}x-\tfrac{1}{8}x^2\pm\tfrac{1}{16}x^3-\ \dots \qquad (1)$$

$$(1\pm x)^{\frac{1}{3}}=1\pm\frac{x}{3}-\frac{1}{9}x^2\pm\frac{5}{81}x^3-\ \dots \qquad (2)$$

Illustrative examples.

1. Find the square root of 37.

Solution. Substituting in (1),

$$\sqrt{37}=(36+1)^{\frac{1}{2}}=6(1+\tfrac{1}{36})^{\frac{1}{2}}$$
$$=6\left(1+\frac{1}{72}-\frac{1}{8\cdot 36^2}+\frac{1}{16\cdot 36^3}\quad\cdot\quad\cdot\quad\cdot\quad\right)$$
$$=6+.08333-.00058+.000008=6.083-.$$

2. Find the cube root of 76.

Solution. Using (2),

$$\sqrt[3]{76}=(64+12)^{\frac{1}{3}}=4\left(1+\frac{3}{16}\right)^{\frac{1}{3}}=4\left(1+\frac{1}{16}-\frac{1}{256}+\frac{5}{12288}\right.$$
$$\left.\cdot\quad\cdot\quad\cdot\quad\right)=4+.25-.015625+.0016=4.236-.$$

Exercise 215

Find the square root of the following numbers, using four terms of (1):

1. 26. 2. 38. 3. 53. 4. 150. 5. 87. 6. 200.

Find the cube root of the following numbers, using four terms of (2):

7. 28. 8. 66. 9. 130. 10. 231.

CHAPTER XXI

CUMULATIVE REVIEW

226. Factoring and fractions.

Exercise 216

Factor the following:

1. $x^3+3x^2-4x-12$.
2. $(x+2)^2(x-5)-(x-3)(x-5)-5(x-5)$.
3. a^2-25b^2-6a+9.
4. $2(a^3+1)-7(a^2-1)$.
5. $a^4+a^2b^2+b^4$.
6. x^4+2x^2+9.

Simplify the following:

7. $\frac{8a^3+b^3}{9a^2-4b^2}\cdot\left(1+\frac{4b}{3a-2b}\right)\div\frac{2a+b}{9a^2-12ab+4b^2}$.
8. $\left[\frac{a^4+a^2+1}{a^3+1}\cdot\frac{(a+1)^2}{a^2-1}\div\frac{a^3-1}{(a-1)^2}\right]-1$.
9. $\frac{x+2a}{x-a}-\frac{x^2+7ax}{a^2-x^2}+\frac{x-2a}{x+a}$.
10. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{n^2}-\frac{1}{m^2}}\div\frac{a^2-b^2}{m^3-n^3}$.
11. $\dfrac{\frac{x+y}{y}+\frac{y}{x+y}}{\frac{1}{x}+\frac{1}{y}}$.
12. $a+\dfrac{a}{a+\frac{1}{a}}=a+\dfrac{a}{\frac{a^2+1}{a}}=a+\frac{a^2}{a^2+1}=?$
13. $x-\dfrac{x}{x-\dfrac{x}{x-\frac{1}{x}}}=x-\dfrac{x}{x-\dfrac{x}{\frac{x^2-1}{x}}}=x-\dfrac{x}{x-\frac{x^2}{x^2-1}}=?$

14. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}}$. Ans. $\dfrac{5}{8}$.

15. $\dfrac{\frac{3}{4}\cdot\frac{3}{4}-1}{\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}-1}$.

16. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}$.

17. $\dfrac{a-2}{a-2-\dfrac{a}{a-\dfrac{a-1}{a-2}}}$.

18. $\dfrac{1}{m-\dfrac{m^2-1}{m+\dfrac{1}{m-1}}}$.

227. Exponents and radicals.

Exercise 217

Find the numerical value of each of the following:

1. $(-64)^{-\frac{2}{3}}$.

2. $16^{-\frac{3}{4}}\div 4^{\frac{3}{2}}$.

3. $9^{\frac{3}{2}}\div 27^{-\frac{2}{3}}$.

4. $\dfrac{16^{-\frac{3}{4}}\cdot 27^{\frac{4}{3}}}{9^{\frac{3}{2}}\cdot 64^{-\frac{2}{3}}}$.

5. $3^{-2}\cdot\dfrac{7}{12^{-1}}$.

6. $(6^{-2}\cdot 4^{-2}\cdot 3^3)\div(2^{-8}\cdot 9^{-2})$.

7. $8^0\cdot 9^{-1}\cdot 3^{-2}\div 27^{-1}$.

Solve for a:

8. $27^a\cdot 9=3^{2a}$.

9. $8^{2a}\cdot 4^{3a}=16^3$.

10. $(9^a)^{2a}=\dfrac{3}{27^{-a}}$.

11. $36^{2a}=\dfrac{(6^a)^a}{6^{-4}}$.

12. $a^{-\frac{2}{3}}=8$.

13. $a^{-\frac{3}{4}}=125$.

14. $\dfrac{\sqrt[3]{a^{\frac{4}{5}}}}{\sqrt[3]{a^{\frac{1}{5}}}}=\dfrac{\sqrt[5]{5}}{\sqrt[5]{6}}$.

15. $\dfrac{1}{8}\,a^{-\frac{4}{5}}=2$.

Simplify:

16. $(x^{a-1})\,(x^{2-a})^2\,(x^{a-4})^{-1}$.

17. $\left(x^{\frac{a+2}{a+3}}\right)\left(x^{\frac{1}{a+3}}\right)$.

18. $x^{a^3-b^3}\div x^{a-b}$.

19. $\left(x^{\frac{1}{n^2-1}}\right)^{n+1}$.

20. $(a^{\frac{1}{2}}x^{-\frac{1}{2}}\sqrt{ax^{-\frac{1}{4}}\sqrt[3]{x}})^{\frac{1}{2}}$.

21. $\left(\sqrt{x^{a^2}}\div\sqrt[3]{x}\right)^{\frac{3}{3a-1}}$.

22. $\dfrac{a\sqrt[4]{x^3}}{\sqrt{y}\sqrt{a}}\div\dfrac{\sqrt{a^{-2}}}{\sqrt[3]{ax}\cdot x^{-\frac{1}{6}}}$.

23. $\left(m^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{mx^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\left(\sqrt{a^2b^{-1}c\sqrt{b^2c^{-1}a\sqrt{c^2a^{-1}b\sqrt{abc}}}}\right)^{16}$.

25. $x^{\frac{a}{a+b}}\div x^{\frac{b}{a+b}}$.

26. $[\{(x^a)^{-b}\}^{-c}]\div[\{(x^c)^{-a}\}^{b}]$.

27. $(ab)^{x+y}\div a^xb^y$.

28. $[(a^{x+y})^{x-y}\div(a^{y-x})^{x}]$.

29. $\left(x^{\frac{a}{4}}y^{-\frac{c}{3}}z^{ac}\right)^{\frac{12}{ac}}$.

30. $(4^{a+2}+4\cdot4^a)\div(16\cdot4^{a+2})$.

31. $\sqrt{\dfrac{3^{n+2}}{9^{-n}}\div\dfrac{27^n}{3^3}}$.

32. $(a^{x-1})^3(a^{2+x})^3(a^{x-3})^{-2}$.

33. $\dfrac{5^{-2}-3^{-2}}{5^{-3}-3^{-2}}$.

34. $\dfrac{x^{-1}+3a^{-1}}{x^{-3}+27a^{-3}}$.

35. $\dfrac{a^{-4}+a^{-2}+1}{a^{-2}-a^{-1}+1}$.

36. $\dfrac{(.008)^{-\frac{1}{3}}\cdot\sqrt{25}}{(.04)^{-\frac{3}{2}}\cdot(2.25)^{-\frac{1}{2}}}$.

37. $\dfrac{1-x^{-2}b^2}{x^{-1}-x^{-2}b}\cdot\dfrac{x^{-1}b^{-1}}{x^{-1}+b^{-1}}$.

38. $\dfrac{8\cdot2^3\cdot4^{n-5}}{4\cdot2^{2n}\cdot8^2}$.

39. $(a^{5x}-5a^{3x}+10a^x-10a^{-x}+5a^{-3x}-a^{-5x})$
$\div(a^{2x}+a^{-2x}-2)$.

40. $(5\sqrt{5}-\sqrt{7}+9\sqrt{3}+2\sqrt{105})(\sqrt{3}-\sqrt{7}+\sqrt{5})$.

41. $(3\sqrt{\tfrac{2}{3}}-2\sqrt{\tfrac{1}{3}}+10\sqrt{\tfrac{1}{5}})(\tfrac{1}{2}\sqrt{24}+\tfrac{1}{5}\sqrt{125}+\sqrt{108})$.

42. $(\sqrt{a+1}-\sqrt{a-1})(\sqrt{a-1})$.

43. $(\sqrt{a+1}-2)^3$.

Rationalize the denominators:

44. $\dfrac{\sqrt{5}+\sqrt{2}}{(2\sqrt{5}+\sqrt{2})(18+4\sqrt{10})}$.

45. $\dfrac{\sqrt{3-\sqrt{2}}}{\sqrt{2-\sqrt{3}}\sqrt{3+\sqrt{2}}}$.

46. $\dfrac{\sqrt{7}-\sqrt{3}}{2+\sqrt{3}+\sqrt{7}}$.

47. $\dfrac{1}{2+\sqrt{3}+\sqrt{5}+\sqrt{2}}$.

48. $\dfrac{3+\sqrt{2}}{\sqrt{3}+\sqrt{3+\sqrt{2}}}$.

49. $\dfrac{5\sqrt{3}+3\sqrt{2}}{\sqrt{2}+\sqrt{3}}$.

50. $\dfrac{\sqrt{26+8\sqrt{3}}}{\sqrt{6}-\sqrt{2}}$.

51. $\dfrac{2ab}{\sqrt[5]{9a^3b^2}}$.

52. $8\sqrt{16\frac{1}{3}}+2\sqrt{27}-5\sqrt{363}+6\sqrt{\dfrac{100}{3}}$.

53. $\dfrac{1}{1-\sqrt{-2}}$.

54. $\dfrac{2+\sqrt{-2}}{2-\sqrt{-2}}$.

55. $\dfrac{\sqrt{a^3}}{\sqrt{-a^5}}$.

56. $\dfrac{\sqrt{3}+\sqrt{-2}}{1-\sqrt{-1}}$.

57. $\dfrac{\sqrt{-3}-\sqrt{-2}}{\sqrt{-3}+\sqrt{-2}}$.

58. $\dfrac{x-\sqrt{y-z}}{x+\sqrt{z-y}}$.

59. $\dfrac{4\sqrt{-2}+3\sqrt{2}}{2\sqrt{-2}-3\sqrt{2}}$.

60. $\dfrac{3m-2n\sqrt{-1}}{2m-n\sqrt{-1}}$.

Extract the square root of:

61. $a^{-4x}+4a^{-3x}-2a^{-2x}-12a^{-x}+9$.

62. $4a^3+5a^2-11a+4-12a^{\frac{5}{2}}+14a^{\frac{3}{2}}-4a^{\frac{1}{2}}$.

63. $a^5-2a^4x^{-1}+5a^3x^{-2}-6a^2x^{-3}+6ax^{-4}-4x^{-5}+a^{-1}x^{-6}$.

64. $a^6-\dfrac{a^5}{2}+\dfrac{33a^4}{16}-\dfrac{a^3}{2}-\dfrac{a^3}{b}+a^2+\dfrac{a^2}{4b}-\dfrac{a}{b}+\dfrac{1}{4b^2}$.

65. Find to three decimals: $\sqrt{\dfrac{\sqrt{3}+\sqrt{17}}{\sqrt{120}}}$.

66. Find to two decimals: $\sqrt[3]{\sqrt{10}-\sqrt{5}}$.

67. Collect $\dfrac{1}{(3-\sqrt{2})^2}+\dfrac{1}{(3+\sqrt{2})^2}$.

68. Collect $\dfrac{1}{a-\sqrt{a^2-4}}+\dfrac{1}{a+\sqrt{a^2-4}}$.

69. Simplify $\left(\sqrt{\dfrac{a}{b}+\dfrac{b}{a}}-\sqrt{\dfrac{a}{b}-\dfrac{b}{a}}\right)^2$.

70. Collect $\dfrac{\sqrt{a}+\sqrt{b}}{2\sqrt{b}}-\dfrac{\sqrt{a}-\sqrt{b}}{2\sqrt{a}}$.

228. Equations.

Exercise 218

Solve the following equations:

1. $(2a-1)^2-2(a+2)(2a-7)=19-3a.$
2. $\frac{n}{9}(n-3)-\left(\frac{n+1}{3}\right)^2=\frac{2}{3}(9+n).$
3. $\frac{x}{1-2x}+\frac{x+1}{4x^2-1}=\frac{x-2}{1+2x}.$
4. $\frac{5a^2-2a-4}{a^2+2a-3}+\frac{2a+3}{1-a}=\frac{3a-1}{a+3}.$
5. $\left(\frac{x}{2x+1}-1\right)\left(\frac{x}{x+1}+1\right)+\frac{3x-2}{2x-1}=\frac{3}{7}.$
6. $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6.$
7. $(3a^2-5a+1)^2-5(3a^2-5a+1)+6=0.$
8. $\sqrt{x+5}-\sqrt{x}=1.$
9. $\sqrt{x+4}+\sqrt{2x-1}=\sqrt{7x+1}.$
10. $\sqrt{2x+3}+\sqrt{x+1}=\sqrt{7x+4}.$
11. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{7x+4}.$
12. $\sqrt{7+2x}+\sqrt{1-x}=\sqrt{2x+15}.$
13. $\sqrt{2x-3}+\sqrt{x-2}-\sqrt{3x-5}=0.$
14. $\sqrt{2x-3}-\sqrt{x-2}-\sqrt{3x-5}=0.$
15. $-\sqrt{2x-3}+\sqrt{x-2}-\sqrt{3x-5}=0.$
16. $\sqrt{2x-3}+\sqrt{x-2}+\sqrt{3x-5}=0.$
17. $\sqrt{3x-2}+\sqrt{2-x}=\sqrt{2x}.$
18. Write and solve three other examples involving the same radicals as No. 17, but differing in signs. Compare with Nos. 13–16.
19. $\sqrt{3x-5}+\sqrt{2x-3}=\sqrt{5x-8}.$
20. Write and solve three other examples involving the same radicals as No. 19.

21. $\sqrt{x-\sqrt{x^2-3\sqrt{x+4}}}=1.$

22. $\sqrt{x+\sqrt{x^2+3\sqrt{x+4}}}=1.$

23. $\sqrt{5+x}-\dfrac{2}{\sqrt{5-x}}=\sqrt{5-x}.$

24. $\dfrac{\sqrt{3+x}}{\sqrt{3-2x}}-\dfrac{\sqrt{3-2x}}{\sqrt{3+x}}=\dfrac{3}{2}.$

25. $x^2-3x+4-2\sqrt{x^2-3x+5}=2.$

26. $x+x^{\frac{1}{2}}=30.$ 27. $3x^{-1}-5x^{-\frac{1}{2}}=2.$

28. $5a^{\frac{2}{3}}-3a^{\frac{1}{3}}=14.$ 29. $2a^{-\frac{2}{3}}-5a^{-\frac{1}{3}}+2=0.$

Solve the following for x:

30. $2a^2x^2+abx=3b^2.$

31. $abx^2-bx+2ax-2=0.$

32. $anx^2-2ax+3nx=6.$

33. $mx^2-m^2x+mnx-x+m-n=0.$

34. $\left(ax-\dfrac{6}{ax}\right)^2-6\left(ax-\dfrac{6}{ax}\right)+5=0.$

229. Miscellaneous applications.

Exercise 219

1. Given the formulas $i=prt$ and $a=p+prt$, eliminate p and derive a formula for i.

2. Solve $F=32+\dfrac{9C}{5}$ for C.

3. Given the formulas $S=4\pi r^2$ and $V=\frac{4}{3}\pi r^3$, eliminate r and solve for V.

4. Eliminate n from $l=a+(n-1)d$ by the use of $s=\dfrac{n}{2}(a+l)$.

5. Solve $l=ar^{n-1}$ for n in terms of a logarithmic formula.

6. Eliminate a from $s=\dfrac{a-ar^n}{1-r}$ by the use of $l=ar^{n-1}$.

7. Solve the formula $s=\frac{1}{2}gt^2+vt$ for t.

8. Eliminate t from $s=\frac{1}{2}gt^2$ and $v=gt$, and obtain a formula for v in terms of the other letters.

9. Find the roots of $.23x^2-1.21x=2.7$ correct to .001.

10. Find, without logarithms, the value of $(13.7)^{\frac{3}{2}}$.

11. Find, by the use of logarithms, the value of $(.00327)^{-\frac{1}{3}}$.

12. Solve by logs $\sqrt[5]{\dfrac{(2.87^2)\ (.03275)}{1.83^3}}$.

13. Reduce $a^{15}-b^{15}$ to its four prime factors.

14. Eliminate y from $x^2+y=7$ and $x+y^2=11$ and find one root of the resulting equation of the fourth degree by synthetic division. Find the corresponding value of y and check.

15. Solve for x and y:

$$\frac{x-y}{3}\div\left(\frac{x+1}{2}-\frac{y-1}{3}\right)=\frac{2}{3}$$

$$\frac{x-3y}{2}-\frac{2x+y}{3}=8.$$

16. Find four consecutive odd numbers such that the product of the third and fourth exceeds the sum of the squares of the first and second by 13.

17. A has $2.95 in nickels, dimes, and quarters, 20 coins in all. If the number of nickels is one-third the number of dimes and quarters together, find the number of each.

18. A motor-boat requires 6 hours to go 18 miles down stream and return. If the current were one-half as swift, the motor-boat could make the round trip in 4 hours and 48 minutes. Find the rate of the boat and the rate of the current.

19. If a number composed of two digits is multiplied by the digit in units' place, the product is 24 times the sum of the digits. The digit in units' place is 3 more than the digit in tens' place. Find the number.

20. A farmer has in one bin feed composed of 2 parts corn to 3 parts wheat. In another bin he has ground feed composed of 5 parts corn to 3 parts wheat. He wished to obtain 90 pounds of feed half corn and half wheat. How much must he take from each bin?

21. A mechanic determines that if his wages are increased $12\frac{1}{2}$c per hour, it will require 40 hours less time for him to earn $100. What are his present wages?

22. A locomotive engineer whistled for a crossing at a certain distance from it. The sound of the whistle is heard at the crossing 25 seconds before the arrival of the train. If the speed of the train was 80 feet per second and sound travels 1080 feet per second, how far was the train from the crossing when the engineer blew his whistle?

23. Separate 21 into two parts so that one part increased by 50% of itself is to the square of the other part as 2 : 9.

24. Two autos start toward each other from two towns at the same time and meet in 4 hours. One auto travels at a uniform rate of 6 miles per hour faster than the other and requires $3\frac{1}{3}$ hours less time to go the entire distance between the towns. Find the rate of each auto and the distance of one town from the other.

25. A man has two investments, one yielding 3% and the other 4%, from which he derives an annual income of $590. If the investments were interchanged, his annual income would be $600. Find the amount of each investment.

26. The arithmetic mean between two numbers is $37\frac{1}{2}$ and their geometric mean is 36. Find the numbers.

27. A and B together can do a piece of work in half the time required by C to do it alone. A works twice as fast as B. Also B and C can do the work together in $4\frac{4}{5}$ days. In what time can each do the work alone?

28. The ages of A and B are 20 and 13 years, respectively. In how many years will their ages have the ratio 4 : 3?

29. The denominator of a certain fraction exceeds the numerator by 3. If a certain number is added to both terms of the fraction, the value of the fraction is $\frac{5}{6}$. If the same number is subtracted from both terms of the fraction, the value is $7/10$. Find the fraction.

30. It is proved in geometry that one side of a regular decagon inscribed in a circle is a mean proportional between the radius and the difference of the radius and the side. Find the side of a regular decagon inscribed in a circle whose radius is 12 inches.

31. Find the apothem and area of the decagon of No. 30.

32. Solve $d=16t^2$ for t and use the resulting formula to determine the number of seconds required for a body to fall 900 feet.

33. A stone is dropped from the top of Washington Monument, the height of which is 555 feet. In how many seconds will it strike the ground?

34. A stone is dropped from the top of a cliff overhanging a river and 5 seconds later its splash is heard in the water below. If sound travels 1080 feet per second, find the height of the cliff. Ans. 350 feet.

35. An auto starts from a town at 8 A. M., traveling at the rate of 7 miles the first hour, 8 miles the second hour, 9 miles the third hour, and so on. A second auto starts at 9 A. M. and travels at a uniform rate of 12 miles per hour. At what times will the two autos be together?

36. A merchant sells an article for $75 and, computing his percentage of profit on the purchase price, finds that his percentage of profit equals the number of dollars in the purchase price. What was the purchase price?

37. A merchant sells an article for $90 and, computing his percentage of profit on the purchase price, finds that his percentage of profit is 10 less than the number of dollars in the purchase price. Find the purchase price.

38. The perimeter of a right triangle is 48 feet and its area is 96 square feet. Find its legs and hypotenuse.

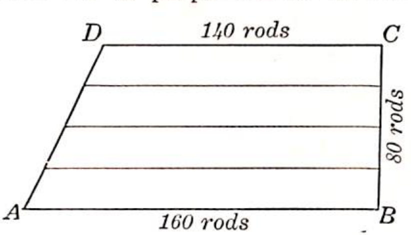
39. The front wheels of a farm wagon make 88 more revolutions per mile than the rear wheels. The front wheels of a farm truck are 2 feet less in circumference and the rear wheels 3 feet less than the corresponding wheels of the wagon, and the front wheels of the truck also make 88 more revolutions per mile than its rear wheels. Find the circumferences of both sets of wheels.

40. It has been shown that π is four times the fraction

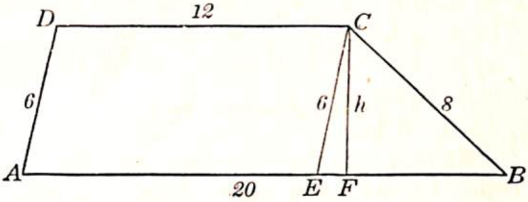
$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{16}{2+\cfrac{25}{2+}}}}}\quad\cdots\cdot$$

Now the value of π, correct to 6 decimals, is 3.141592. How much does four times the value of the fraction, stopping at $2+\dfrac{64}{2\ \ldots}$, differ from the value of π? How much, stopping at $2+\dfrac{100}{2\ \ldots}$?

41. A farmer has a field in the shape of the trapezoid of the accompanying figure. Its dimensions are $AB=160$ rods, $BC=80$ rods, and $CD=140$ rods. AB and CD are parallel and BC is perpendicular to both. He wishes to divide it equally among his four sons by fences parallel to AB and CD. How far apart will these fences be, and what frontage will each son have on the road AD?



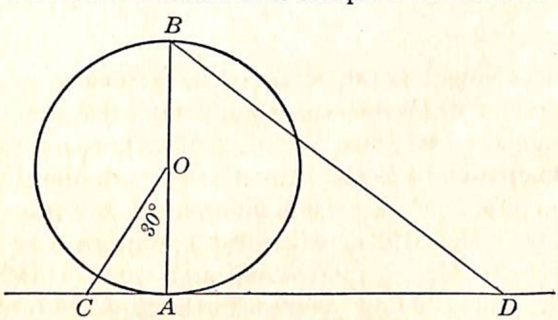
42. Find the area of a triangle whose sides are 12, 16, and 24 inches, respectively.



The area of a trapezoid all of whose sides are known as in the accompanying figure, can be found if h, the distance between the parallel sides, is considered as the altitude of the triangle BCE. The value of h can be found by solving the equations: $x^2+h^2=36$, and $(8-x)^2+h^2=64$, where $x=EF$. Explain.

43. Find the area of the trapezoid whose parallel sides are 24 and 36 feet and whose non-parallel sides are 8 and 10 feet, respectively.

44. The value of the continued fraction $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dotsb}}}$ is one root of $\frac{1}{x}=\frac{x}{1-x}$. Solve the equation, simplify the continued fraction and compare the results.



45. In the accompanying figure, CD is tangent to the circle whose center is O at A, one end of the diameter AB. $\angle AOC=30°$ and CD is constructed equal to $3r$.

Show that $CA=\frac{1}{3}r\sqrt{3}$. Then $AD=3r-\frac{1}{3}r\sqrt{3}$.

Compare the length of BD with the length of the semicircle.

CHAPTER XXII

SUPPLEMENTARY TOPICS

INTERPRETATION OF IMAGINARIES

230. Conjugate imaginaries. Complex numbers that differ only in the sign of the imaginary term are called **conjugate imaginaries.** Thus $5+2\sqrt{-1}$ and $5-2\sqrt{-1}$ are conjugate. The typeforms are $a+bi$ and $a-bi$.

Now $(a+bi)+(a-bi)=2a$, $(a+bi)-(a-bi)=2bi$, and $(a+bi)(a-bi)=a^2+b^2$. Evidently the sum and the product of a pair of conjugate imaginaries are real numbers, but the difference is an imaginary number.

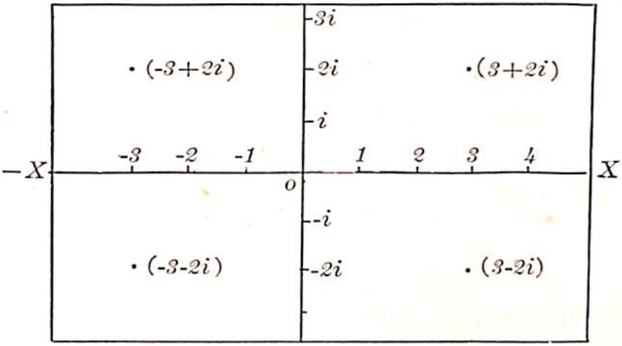
If a complex number is one root of an equation of second or higher degree, then its conjugate is a second root.

Given two complex numbers not conjugate, evidently their sum, difference, product, and quotient are complex numbers unless the imaginary term becomes zero.

Since the square of a complex number is a complex number, the square root of any complex number may be expressed as a complex number following the plan of § 179, except that the rational part is the difference of two factors of m.

231. Graph of a complex number. Since the symbol $\sqrt{-1}$, or i, can be interpreted as an operator that turns a real number through an angle of 90° (see § 180), it may be used to indicate direction on a plane as the signs $+$and $-$ are used to indicate direction on a line. For $+3$ is interpreted to mean a distance of 3 units to the right of an agreed 0 point on a line in contrast with -3 which is interpreted as 3 units to the left of the point. Then $3i$ may mean a distance of 3 units measured upward on a line which is perpendicular to the first line at the point O, and $-3i$ a distance downward on the same perpendicular line.

A complex number may be represented graphically as a point on a plane; for $3+2i$ is to be found by moving 3 units to the right of O then 2 units at right angles upward to the



point indicated on the figure. Similarly $3-2i$, $-3+2i$, and $-3-2i$ may be located as on the figure.

Each complex number may be represented by one and only one point on the plane, and each point on the plane, not on an axis, locates one complex number.

Exercise 220

Graph each of the following complex numbers:

1. $1-2i$. **2.** $-2+i$. **3.** $-3+3i$.

4. $-1-2i$. **5.** $-2-2i$. **6.** $-1+i\sqrt{3}$.

7. $-\sqrt{2}-i\sqrt{3}$. **8.** $\sqrt{3}+i\sqrt{2}$. **9.** $-2-2i\sqrt{2}$.

10. $-1-2i\sqrt{2}$. **11.** $2\sqrt{2}-i\sqrt{3}$. **12.** $-\sqrt{3}+2i\sqrt{3}$.

232. Graphical addition and subtraction of complex numbers.

If the point A, $(3+i)$, and the point B, $(1+3i)$, are each connected with O and the parallelogram completed as in the

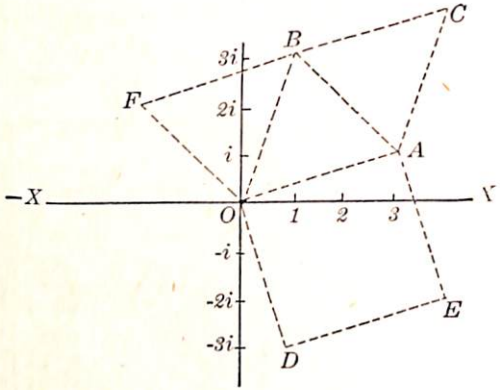


figure with A, O, and B as three consecutive vertices, then the fourth vertex, C, is the graph of the complex number $4+4i$, or the sum of $3+i$ and $1+3i$.

Similarly, the point E, $(4-2i)$, is the fourth vertex of the parallelogram of A, $(3+i)$, O, and D, $(1-3i)$.

The difference between two complex numbers may be represented as the fourth vertex of a parallelogram, one diagonal of which is the line formed by connecting the point of the minuend with O.

If we wish to find the point determined by $(1+3i)-(3+i)$, we complete the parallelogram O, A, $(3+i)$, and B, $(1+3i)$, which locates the point F, $(-2+2i)$.

Exercise 221

Represent graphically the following:

1. $(3-2i)+(1+4i)$.

2. $(-2-3i)+(1-i)$.

3. $(-3+i)-(-2-2i)$.

4. $-(-2-2i)+(-2+2i)$.

5. $(4+4i)-(2-3i)$.

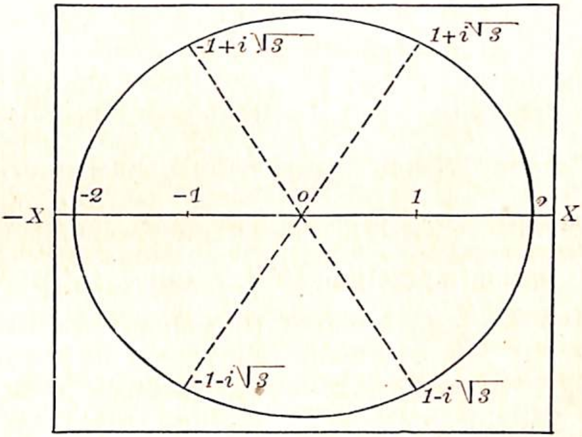
6. $(\sqrt{2}-2i)+(2-i\sqrt{3})$.

7. $(2\sqrt{2}-2i)-(2+2i\sqrt{3})$.

8. $-(2\sqrt{3}-i)+(\sqrt{3}-2i\sqrt{2})$.

233. Graphing roots of numbers. The point as the graph of a complex number furnishes a means for representing the roots of numbers.

The six sixth roots of 64 are located on the circumference of a circle, radius 2, and are 60° of arc apart, as in the figure.



Solution.

$x^6-64=0.$

Factoring, $(x-2)\ (x^2+2x+4)\ (x+2)\ (x^2-2x+4)=0.$

The two real roots are $x=2$ and $x=-2.$

Solving $x^2-2x+4=0,$

gives $x=1+\sqrt{-3}$, and $x=1-\sqrt{-3}.$

Solving $x^2+2x+4=0,$

gives $x=-1+\sqrt{-3}$, and $x=-1-\sqrt{-3}.$

These roots in the order of their location on the circle are $x=2$, $x=1+\sqrt{-3}$, $x=-1+\sqrt{-3}$, $x=-2$, $x=-1-\sqrt{-3}$, and $x=1-\sqrt{-3}.$

Similarly, locate the cube roots and the square roots of 64.

Exercise 222

Solve and locate all of the roots in each of the following:

1. $x^3-27=0$.
2. $x^4-16=0$.
3. $x^3+8=0$.
4. $x^6-729=0$.
5. $8x^3+27=0$.
6. $x^6+64=0$.

THE INDETERMINATE FORMS

234. Zero in multiplication. It is a rule of arithmetic that if one or more factors are zeros the product is zero; i.e., $7\cdot 0\cdot 5=0$. The same rule holds in algebra for $a\cdot b\cdot 0=0$.

235. Zero in division. Occasionally such forms as $\frac{0}{a}$, $\frac{a}{0}$, and $\frac{0}{0}$ are met with in algebra, especially in checking an equation or in evaluating a fraction. For instance, if, in evaluating the fraction $\frac{x+2}{x-2}$, we let $x=2$, the value of the fraction becomes $\frac{4}{0}$. If we let $x=-2$, its value becomes $\frac{0}{4}$. If, in evaluating the fraction $\frac{x^2-4}{x-2}$, we let $x=2$, its value becomes $\frac{0}{0}$.

That the form $\frac{0}{a}=0$ is evident if it is recalled that a product is zero only when a factor is zero, and that the dividend (the numerator of the fraction) is the product of the divisor (the denominator) and the quotient (the value of the fraction). Since the divisor is not zero the quotient must be. Therefore $\frac{0}{a}=0$.

The forms $\frac{a}{0}$ and $\frac{0}{0}$ can have no meaning in real numbers for division by zero cannot be allowed. But there are occasions when it is necessary to find some interpretation for $\frac{a}{0}$.

236. Interpretation of $\frac{a}{0}$. If 5 is divided by .1, the quotient is 50. If 5 is divided by .01 the quotient is 500, if by .001, the quotient is 5000, if by .0001, the quotient is 50000, etc. In equation form this becomes $\frac{5}{.1}=50$, $\frac{5}{.01}=500$, $\frac{5}{.001}=5000$, $\frac{5}{.0001}=50000, \ldots$

Evidently, as the denominator decreases in size the value of the fraction increases, and, as the denominator becomes exceedingly small, the value of the fraction becomes exceedingly large.

In this series of operations the numerator, 5, remains unchanged throughout the discussion but the denominator changes. The numerator is said to be a **constant** and the denominator a **variable.**

A **constant** is a number that retains the same value throughout a particular mathematical discussion. A **variable** is a number that changes its value in the discussion.

The **limit of a variable** is that constant, the difference between which and the variable may be made to become and remain less than any assigned positive quantity, however small. The variable is said to approach this constant as its limit and the symbol $\doteq$ is used throughout mathematics to indicate this relation. Such an expression as $x \doteq a$ is to be read "x approaches a as its limit."

If the series of fractional equations $\frac{5}{.1}=50$, $\frac{5}{.01}=500$, $\frac{5}{.001}=5000$, $\frac{5}{.0001}=50000$, $\frac{5}{.00001}=500000, \ldots$ is continued indefinitely in the same manner, evidently the denominator of the left member becomes smaller and smaller and may be made less than any assigned quantity, however small, that is, the denominator $\doteq 0$. At the same time the value of the fraction (the right member) will become larger and larger and will become and remain greater than any positive number which may be assigned. This necessitates a new definition and a new symbol.

If a number may become and remain greater than any positive number that may be assigned, it is said to become infinitely large or to become **infinite**; which means unbounded or unlimited. The symbol ∞ is called **infinity.** It is not the symbol for some number but it is the symbol that the value of the variable exceeds all bounds.

Now in interpreting $\frac{a}{0}$ if we replace $\frac{a}{0}$ by $\frac{a}{x}$ and consider the value of $\frac{a}{x}$ as $x \doteq 0$, evidently $\frac{a}{x}$ increases indefinitely and as $x \doteq 0$, $\frac{a}{x}$ becomes ∞.

Therefore $\frac{a}{0}$, where a is a constant, is said to have the value ∞ which is equivalent to the following:

Principle. *If the numerator of a fraction remains a constant, while the denominator approaches zero, the value of the fraction becomes* ∞.

The value of the fraction $\frac{x+2}{x-2}$ for $x=2$ is ∞.

237. Interpretation of $\frac{a}{\infty}$.

If we consider the series of fractions $\frac{5}{10}, \frac{5}{100}, \frac{5}{1000}, \frac{5}{10000}, \frac{5}{100000}, \cdots$, evidently, as the denominator increases indefinitely in the same manner, the value of the fraction decreases indefinitely.

Replacing $\frac{a}{\infty}$ by $\frac{a}{x}$ and assuming that $x \doteq \infty$ we have $\lim\limits_{x \doteq \infty} \frac{a}{x} = 0$, which is to be read "the limit of $\frac{a}{x}$ as x approaches ∞ is zero."

Therefore $\frac{a}{\infty}$, where a is a constant, is said to have the value 0 which is equivalent to the following:

Principle. *If the numerator of a fraction remains a constant while the denominator approaches infinity, the value of the fraction approaches zero.*

An expression involving a single variable, that becomes indeterminate for a certain value of that variable, may be interpreted on the plan of the following **illustrative examples:**

1. Find the value of $\frac{x^2-4}{x-2}$ when $x=2$. Evidently, when $x=2$ the value of the given fraction is $\frac{0}{0}$.

But for x not equal to 2, $\frac{x^2-4}{x-2} = x+2$. Now, $\lim\limits_{x \doteq 2} (x+2) = 4$. Therefore, when $x=2$, we may give to $\frac{x^2-4}{x-2}$ the value 4.

2. Find the value of $\frac{4x^2+3x-1}{x^2+1}$ as $x \doteq \infty$.

For any finite value of x other than 0,

$$\frac{4x^2+3x-1}{x^2+1} = \frac{4+\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}.$$

$$\text{Now, } \lim_{x \doteq \infty} \left\{ \frac{4+\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}} \right\} = \frac{4+0-0}{1+0} = 4.$$

Therefore, when $x \doteq \infty$, give to $\frac{4x^2+3x-1}{x^2+1}$ the value 4.

Exercise 223

Find the value of each of the following as $x \doteq 0$:

1. $\frac{5}{x}$. Ans. ∞ (See principle).

2. $\frac{5}{x^2}$.

3. $\frac{5}{\frac{2}{x}}$. Ans. 0.

4. $\frac{x}{1-x}$.

5. $\frac{x^2}{2x}$.

6. $\frac{3x}{x-2}$.

Find the value of each of the following as $x \doteq \infty$:

7. $2x^3$.

8. $\frac{a}{x}$.

9. $2+\frac{2}{x}$.

10. $\frac{x}{x^2}$.

11. x^2+2.

12. $(x+2)^2$.

13. $\left(\frac{1}{x}+2\right)^2$.

Interpret each of the following:

14. $\underset{x \doteq 3}{\text{limit}}\left(\frac{x^2-9}{x^2-7x+12}\right)$. Ans. -6.

15. $\underset{x \doteq 2}{\text{limit}}\left(\frac{x^2+5x-14}{x^2-3x+2}\right)$.

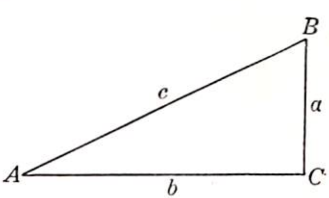
16. $\underset{x \doteq \infty}{\text{limit}}\left(\frac{x^2-2x+4}{x^2+2x+4}\right)$.

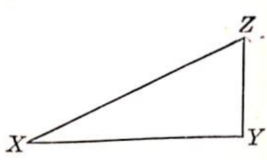
17. $\underset{x \doteq 5}{\text{limit}}\left(\frac{x^2+x-30}{x-5}\right)$.

18. $\underset{x \doteq \infty}{\text{limit}}\left(\frac{x+7}{x^2}\right)$.

TRIGONOMETRIC RATIOS

238. Trigonometry deals with the ratios obtained from the lengths of the sides of right triangles. The student will find when he comes to a more complete study of the subject that the development of its formulas requires a large use of algebra.





Given any two right triangles such as ACB and XYZ, as in the accompanying figures, with $\angle A = \angle X$, we know from a theorem of geometry that the two are mutually equiangular and therefore similar.

$$\therefore \frac{BC}{AB}=\frac{ZY}{XZ}, \frac{AC}{AB}=\frac{XY}{XZ}, \text{ and } \frac{BC}{AC}=\frac{ZY}{XY}. \quad \text{(Def.)}$$

Will these ratios remain equal for all rt. $\triangle$s having an acute $\angle$ equal to $\angle A$?

The constant ratio BC/AB is the **sine** of $\angle A$. That is, the **sine** of an acute angle of a right triangle is the ratio of the side opposite it and the hypotenuse.

The constant ratio AC/AB is the **cosine** of $\angle A$. That is, the **cosine** of an acute angle of a right triangle is the ratio of the side adjacent and the hypotenuse.

The constant ratio BC/AC is the **tangent** of $\angle A$. That is, the **tangent** of an acute angle of a right triangle is the ratio of the opposite and adjacent sides.

Using a, b, and c for the lengths of the three sides,

$$\sin A=\frac{a}{c}, \cos A=\frac{b}{c}, \tan A=\frac{a}{b}.$$

From the definitions it will be observed,

$$\sin B=\frac{b}{c}, \cos B=\frac{a}{c}, \tan B=\frac{b}{a}.$$

Exercise 224

1. Show that the sine of an acute angle is equal to the cosine of its complement.

2. Show that the sine of an acute angle is less than 1. Is this true for the cosine? The tangent?

3. If two acute angles are unequal which will have the greater sine? The greater cosine? The greater tangent?

4. What is the acute angle whose sine equals its cosine? What is the tangent of this angle?

5. Construct an equilateral triangle and one of its altitudes. Using the resulting right triangles, show

$$\sin 30° = \frac{1}{2} = .5,\ \cos 30° = \frac{\sqrt{3}}{2} = .866,\ \tan 30° = \frac{1}{\sqrt{3}} = .577,$$

$$\sin 60° = \frac{\sqrt{3}}{2} = .866,\ \cos 60° = \frac{1}{2} = .5,\ \tan 60° = \sqrt{3} = 1.732.$$

6. Construct an isosceles right triangle and show,
$\sin 45° = .707$, $\cos 45° = .707$, $\tan 45° = 1$.

Note. The trigonometric ratios on page 405 are correct to within .0005. They will be used in solving right triangles having given any side and either acute angle or any two sides. The lettering for the following exercises is the same as that for the first figure on page 402.

7. Given $A = 25°$, $c = 30$ in., find B, a, and b. Solution: $\sin A = a/c$, or $.423 = a/30$. Whence $a = 30(.423) = 12.69$ (in.) $B = 65°$ (Why?) $\sin B = b/c$. Whence $b = 27.18$ in.

8. Given $a = 15$ in., $b = 24$ in., find A, B, and c. Solution: $\tan A = a/b = 15/24 = .625$. By referring to the table we find the angle whose tangent is .625 is 32°. $\therefore A = 32°$ and $B = 58°$. Find c.

9. Given $B = 38°$, $a = 17$ in., find A, b, and c.

10. Given $b = 18$ ft., $c = 36$ ft., find A, B, and a.

11. Given $A = 23°$, $c = 75$ in., find B, a, and b.

12. Given $B = 43°$, $b = 6.23$ in., find A, a, and c.

13. When the sun is 40° high the shadow of a certain tree is 37 feet long. Find the height of the tree.

14. In order to determine the width of a river, a base line AC 100 feet long is measured along one bank. A point B is found on the opposite bank so that $\angle ACB$ is a right angle. If $\angle BAC$ is 73°, how wide is the river?

15. In order to determine the height of a tower CB, a base line CA is measured along the ground 150 feet long. The $\angle BAC$ is found to be 49°. How high is the tower?

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.9976	.070	14.301
42°	.669	.743	.900	87°	.9986	.052	19.081
43°	.682	.731	.933	88°	.9994	.035	28.636
44°	.695	.719	.966	89°	.9998	.017	57.290
45°	.707	.707	1.000	90°	1.000	.000	......

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260–	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780–
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826–
7	49	343	2.646–	1.913–	57	3,249	185,193	7.550–	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.616–	3.871–
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915–
11	121	1,331	3.317–	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958–
13	169	2,197	3.606–	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742–	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021–
16	256	4,096	4.000	2.520–	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061–
18	324	5,832	4.243–	2.621–	68	4,624	314,432	8.246	4.082–
19	361	6,859	4.359–	2.668	69	4,761	328,509	8.307–	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367–	4.121
21	441	9,261	4.583–	2.759–	71	5,041	357,911	8.426	4.141–
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796–	2.844–	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718–	4.236–
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.291	3.037–	78	6,084	474,552	8.832–	4.273–
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309–
31	961	29,791	5.568–	3.141	81	6,561	531,441	9.000	4.327–
32	1,024	32,768	5.657–	3.175–	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745–	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240–	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.397–
36	1,296	46,656	6.000	3.302–	86	7,396	636,056	9.274–	4.414
37	1,369	50,653	6.083–	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381–	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465–
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487–	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498–
42	1,764	74,088	6.481–	3.476	92	8,464	778,688	9.592–	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644–	4.531–
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547–
45	2,025	91,125	6.708	3.557–	95	9,025	857,375	9.747–	4.563–
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579–
47	2,209	103,823	6.856–	3.609–	97	9,409	912,673	9.849–	4.595–
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950–	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642–

INDEX

ANSWER BOOK TO SECOND COURSE IN ALGEBRA

Exercise 120

1. $a + b,\ a - b,\ ab,\ \frac{a}{b}$;

   $3x + 2y,\ 3x - 2y.\ 6xy,\ \frac{3x}{2y}$;

   $abc + xyz,\ abc - xyz,\ abcxyz,\ \frac{abc}{xyz}$.

2. $\frac{xy}{a+b}$.
3. 36; 13; 35; 125.
4. See § 27.
5. See § 34.
6. See § 35.
7. $-abc$. See § 39.
8. 2160.
9. $-4a^2b^2c^2$. See § 41.
10. $-a^6b^5$.
11. $\frac{a}{b}$; $\frac{-2a}{b}$.
12. $\frac{x}{y}$. $-2x$.
13. $-3a + b$; $-2b + c$; $-2x + 3y - 5z$.
14. $a - 2b = 3c - 3b$.
15. $x - 3a = \frac{y}{2c}$.

Exercise 121

1. $x^3 - 5x^2 + 11x + 12$.
2. $-4x^2 + xy - 2y^2$.
3. $a^3 + 8a^2 - 8a - 4$.
4. $a - b + c - d + e$.
5. $6x^4 - 7x^3 - 32x^2 + 32x + 21$.
6. $6n^6 - 9n^5 + 16n^4 - 36n^3 + 17n^2 - 28n + 24$.

7. $x^8 + x^4y^4 + y^8$.
8. $-n^2 + 10n + 6$.
9. $2a^2 + 21a + 45$.
10. $n - 1$
11. $2n^2 + 3n - 5$.
12. $x + y + z$.
13. $a^2 - ab + b^2 + ac + bc + c^2$.

Exercise 123

I. 1. $4a^2b^2(3b - 4a + 7ab)$.
2. $7mn(mn - 2m^2 + 3n^2)$.
3. $17x^3y^3(x^2y^2 - 2xy + 3)$.
4. $13a^5b^5(2a^2 - 3b^2 + 1)$.
5. $11a^4b^5c^6(1 + 2abc - 3a^2b^2c^2 + 4a^3b^3c^3)$.

II. 1. $(n - 4)(n - 4)$.
2. $(x + 3)(x + 3)$.
3. $(4a - 3b)(4a - 3b)$.
4. $(5m + 4n)(5m + 4n)$.
5. $(x^2 + y^2)(x^2 + y^2)$.
6. $(3x^3 + 2y^3)(3x^3 + 2y^3)$.

III. 1. $(x - y)(x + y)$.
2. $(5a - 6)(5a + 6)$.
3. $(6mn^2 - 7x^2y^3)(6mn^2 + 7x^2y^3)$.
4. $(x - y - z)(x - y + z)$.
5. $(2a + 3b - 5)(2a + 3b + 5)$.

IV. 1. $(n - 7)(n - 3)$.
2. $(x - 5)(x + 4)$.
3. $(m - 8n)(m - 3n)$.
4. $(a + 4b)(a + 5b)$.
5. $(r - 8)(r + 5)$.
6. $(k + 12)(k - 5)$.

V. 1. $(3a - 2)(2a - 3)$.
2. $(3x - 2y)(2x + 3y)$.
3. $(7n + 4)(n - 3)$.
4. $(2r + 3s)(2r + s)$.
5. $(3mn - 4)(4mn - 3)$.
6. $(k^3 - 6)(6k^3 - 1)$.

VI. 1. $(a - b)(2x + 3m)$.
2. $(a^2 + b^2)(m^2 - n)$.
3. $(a^3 + x^2)(a^2 - x)$.
4. $(2a + 3b)(3x - 2y)$.
5. $(a + b + c)(x - y)$.

VII. **1.** $7(x-y)(x-2y)$. **3.** $(m-n)^2(m+n)^2$.
2. $a(a-6)(a-3)$. **4.** $(a-b)(a+b)(a^2+b^2)$.
5. $(3x-2)(3x+2)(9x^2+4)$.
6. $a(a-3)(a+3)(a-2)(a+2)$.
7. $(x-y)(x+y)(3x^2+2y^2)$.
8. $(a-b)(a+b)(m-n)(m+n)$.
9. $(a-2)(a+2)(a-5)$.
10. $(2m-n)^2(2m+n)^2$.
11. $4(x-2)^2$.
12. $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$.
13. $(a-b-x-y)^2(a-b+x+y)^2$.
14. $2(2a-b)^2(2a+b)^2$.

Exercise 124

1. $(a-2y)(a+2y)$. **4.** $2x(2x-3y)(2x+3y)$.
2. $a^2(a-2)(a+2)$. **5.** $(8-n^2)(8+n^2)$.
3. $ab(a-b)(a+b)$. **6.** $(a-4)(a+2)$.
7. $(3x-2y)(2x-3y)$.
8. $(a-2)(a+2)(4a^2+1)$.
9. $(a-1)(3a+2)$. **10.** $3x(x-y)^2$.
11. $(a+b-3)(a+b+3)$.
12. $(m+n-a+b)(m+n+a-b)$.
13. $(x-y)(x-y-1)$ **16.** $(a-b-c)(a+b+c)$.
14. $a^3(a-2)^2$. **17.** $(x-y)(x+y+1)$.
15. $5(n-3)(n-2)$. **18.** $(x-y)(x^2y+xy^2+1)$.
19. $(a-b+c)(a+b-c)$.
20. $(5-x+y)(5+x-y)$.
21. $(4-a+b)(4+a-b)$.
22. $(4a+3y)(a+2y)$. **23.** $(a^2-5)(a^2+4)$.
24. $3(2a-b)(2a+b)(4a^2+b^2)$.
25. $(a-2)(a+2)(a-1)(a+1)$.

26. $(a+b)(1+a-b)$. 28. $(a+b-9)(a+b+9)$.
27. $(a+b)(x+y)$. 29. $(3a-5)(3a-2)$.
30. $(5-a)(5+a)(2-a)(2+a)$.
31. $(ab-cd)^2$. 34. $3yz(x-yz)^2$.
32. $(m-n)(8m-3n)$. 35. $3(x^2-3)(3x^2+4)$.
33. $(a+b)(a-b+1)$. 36. $(a-b-c)^2$.
37. $(x+y-5a+5b)(x+y+4a-4b)$.

Exercise 125

1. $x=5$.
2. $n=-5$.
3. $x=-3$.
4. $n=\frac{1}{2}$.
5. $x=a$.
6. $x=4m$.
7. $x=2a$.

Exercise 126

1. $x=3,\ x=-2$.
2. $x=0,\ x=1$.
3. $a=0,\ a=4,\ a=1$.
4. $x=4,\ x=-3$.
5. $x=2,\ x=-\dfrac{1}{9}$.
6. $a=\dfrac{4}{3},\ a=\dfrac{3}{4}$.
7. $x=2,\ x=\frac{1}{2}$.
8. $x=1,\ x=-\frac{1}{4}$.
9. $x=2,\ x=\frac{1}{6}$.
10. $x=4,\ x=-\frac{1}{3}$.
11. $a=2,\ a=-1$.
12. $n=2,\ n=\frac{1}{3}$.

Exercise 127

1. $4a^2b^2$.
2. $25m^2n^2$.
3. $a(a-b)$.
4. $x-3$.
5. $x-y$.
6. $a-1$.
7. $x+5$.
8. 5040.
9. $60a^3b^3$.
10. $a(x+y)^2(x-y)$. 11. $(x-3)(x+3)(x+2)$.
12. $(3a+2b)(3a-2b)(a-b)(a+b)$.
13. $(n-1)(n-2)(n-3)$.

Exercise 128

1. $\frac{2a}{3z}$.
2. $\frac{-2bc}{3a}$.
3. $\frac{x^2}{4z^2}$.
4. $\frac{x+y}{a}$.
5. $\frac{a+2b}{a-b}$.
6. $\frac{3y}{4x}$.
7. $\frac{-1}{m+n}$.
8. $\frac{a+b}{a-b}$.
9. $\frac{-1}{x-2}$.

Exercise 129

The missing terms are:

1. $8x^3y$.
2. $a(a+b)$.
3. $5a(a-2b)$.
4. $-a$.
5. $n-1$.
6. $(x+3)(x-3)$.
7. $15b^2y^2$.
8. $3a(m-n)^2$.
9. $-(x-z)(x+y)$.

Exercise 130

2. $x^2-3x-4-\frac{2}{3x}$.
3. $3a+b+\frac{2b^2}{3a-b}$.
4. $4x^2+6x+9+\frac{54}{2x-3}$.
6. $\frac{2a^2+12a-3}{2a}$.
7. $\frac{a^3-b^3}{a+b}$.
8. $\frac{x^2-5x}{x+3}$.
9. $\frac{2a}{a-b}$.
10. $\frac{2y}{x-y}$.

Exercise 131

1. $\frac{x-y}{y}$.
2. $\frac{a^2-b^2}{ab}$.
3. $\frac{bc+ac+ab}{abc}$.
4. $\frac{5x+y}{(x-y)(x+y)}$.
5. $\frac{7x}{12}$.
6. $\frac{2a^2+2b^2}{(a+b)(a-b)}$.
7. $\frac{m^2}{n(m-n)}$.
8. $\frac{x-12}{(x+3)(x-3)(x-2)}$.
9. 0.

11 $\frac{3x+14}{(x-2)(x+2)}$.

12. $\frac{5x+4y}{(x-y)(x+y)}$.

Exercise 132

1. $\frac{7}{15}$.
2. 1.
3. $\frac{4b}{45ax^2y}$.
4. $\frac{24ay}{25x^2}$.
5. $\frac{n+3}{n-3}$.
6. $\frac{b-c}{a-b}$.
7. $\frac{(xy-1)(xy+2)}{(xy+3)(xy-2)}$.
8. $-\frac{(x-y)(x+y)}{(a-b)^2(x^2+y^2)}$.
9. $\frac{(a-b-c)^2}{(a+b-c)^2}$.
10. $\frac{(a-b)(2a+b)}{(3a-4b)(2a+3b)}$.

Exercise 133

1. $x=12$.
2. $x=20$.
3. $y=-24$.
4. $n=17$.
5. $m=-4$.
6. $n=0$.
7. $x=4a$.
8. $x=\frac{b-a}{2b}$.
9. $x=\frac{a+m+n}{2}$.

Exercise 134

1. $x = 3,\ y = -2.$
2. $x = 2,\ y = -2.$
3. $a = 5,\ b = -3.$
4. $x = 0,\ y = 2.$
5. $m = -2,\ n = -2.$
7. $x = b,\ y = a.$
8. $x = \frac{bk - cn}{bm - an},\ y = \frac{cm - ak}{bm - an}.$

Exercise 135

1. $x = 5,\ y = 0;\ x = 0,\ y = 5.$
2. $x = 5.\ y = 2;\ x = -2,\ y = -5.$
3. $x = 5,\ y = 3.$
4. $x = 4,\ y = -4;\ x = -3,\ y = 3.$
5. $x = 3,\ y = 2;\ x = -2\frac{4}{5},\ y = 4\frac{9}{10}.$
6. $x = 5,\ y = 4;\ x = -2,\ y = -3.$

Exercise 136

1. $15 - x.$
2. 9, 4.
3. $x,\ x + 1,\ x + 2,\ x + 3;\ x + 1,\ x + 3,\ x + 5.$
4. 10, 11, 12.
5. 12, 14, 16, 18.
6. 30, 32, 34.
7. 5, 6, 7, 8.
8. 14, 15, 16.
9. 7, 11, 24.
10. 16, 31.
11. 19.
12. 11 dimes.
13. 20 in.; 13 in.
14. $8,\ -3,\ 2.$
15. $11°,\ -4°,\ 4°.$
16. John has 25¢; James has 15¢.
17. 80°, 40°, 60°.
18. A, 60 years; B, 30 years.
19. 7, 23.
20. 41.
21. 36, 12.
22. 32, 24.
23. 11.
24. 24 persons.
25. $27\frac{3}{11}$ min. past 5 o'clock.
26. $10\frac{10}{11}$ min. past 8 o'clock.

27. $10\frac{10}{11}$ min. past 5 o'clock;

$43\frac{7}{11}$ min. past 5 o'clock.

28. 10, 6.

29. 20 dimes, 32 quarters.

30. Apple, 3¢; orange, 5¢.

31. 140°, 120°, 100°.

32. 4 mi. per hr.

33. $n = \frac{l - a + d}{d}$ $a = l - (n - 1)d$, $d = \frac{l - a}{n - 1}$.

34. $b = \frac{2A - aB}{a}$.

35. Auto, 24 mi. per hr.; train, 40 mi. per hr.

36. 10, 6.

37. 6, 2; -2, -6.

38. 6 in., 8 in., 10 in.

39. 9 in., 5 in.

40. 12 in., 7 in.

41. 20 rods, 12 rods.

42. 120 rods, 40 rods.

Exercise 137

1. -4.
2. 0.
3. 2.
4. 8.
5. 24.
6. $32\frac{5}{16}$.
7. -8.
8. -16.
9. 8.

Exercise 138

1. 54.
2. $7x + 2y + 6$.
3. $18x - 32y - 16z$.
4. $4x - 5y + 2z$.
5. $2x - 7y - 6w + 3z$.
6. $2m - 2n$.
7. $6x + 8y$.
9. $3x - 3$.
10. $2a - 12b + 9c$.
12. $x^2 - 2ax + a^2 - (y^2 - 2yz + z^2)$.
13. $9 - 6a + a^2 - (x^2 + 2xy + y^2)$.
14. $m^2 - 2mn + n^2 - (4a^2 - 12a + 9)$.
15. $25x^2 - 20bx + 4b^2 - (25a^2 + 60ay + 36y^2)$.

16. $x^6 - 4x^5 + 4x^4 - (4x^2 - 4x + 1)$.

19. $x^2 - x(a + b + c) + abc$.

20. $x^3 - x^2(a + b + c) + x(ab + ac + bc) + abc$.

Exercise 139

1. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1$. **3.** $a^2 - 2a + 1$.

2. $m^6 + 5m^4 - m^2 - 25$. **4.** $m^2 + 5m + 7$.

5. $2x^5 - 18x^4 + 39x^3 - 25x^2 + x + 1$.

6. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.

7. $a^2 - a + 1$; $a^2 + a + 1$.

8. $-x^7 + x^6 + 3x^5 - 2x^4 - 2x^3 - 3x^2 + 3x + 3$.

9. $m^3 - 2m^2 - m + 1$. **10.** $m^2 - 2mn + 4n^2$.

11. $m^4 + m^3n + m^2n^2 + mn^3 + n^4$;

$$m^4 - m^3n + m^2n^2 - mn^3 + n^4 - \frac{2n^5}{m + n}.$$

12. $m^5 + n^5$.

Exercise 140

1. $a^2 - 4$: $a^2 - a - 6$. **3.** $x^2 - 2x + 2$. **4.** $a^2 - 5a + 3$.

2. $x^2 + x - 1$; $x^2 + 2x - 1 - \frac{1}{x + 1}$.

5. $x^3 + 5x^2 + 5x + 14 + \frac{24}{x - 2}$.

6. $m^3 + m^2 + 7m + 5$. **8.** $8a^3 + 12a^2 + 18a + 27$; $8a^3 - 12a^2 + 18a - 27$.

7. $4x^2 - 6x + 9$.

9. $2(x - y) - 5$. **10.** $x^2 - 2x + 3$.

Exercise 141

1. $m^2 + n^2 + p^2 + 2mn + 2mp + 2np$.

2. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.

3. $4x^2 + 9y^2 + 4z^2 - 12xy - 8xz + 12yz$.

4. $9m^2 + 4n^2 + 9a^2 + 12mn + 18am + 12an$.

5. $x^2 + y^2 + m^2 + n^2 + 2xy - 2mx - 2nx - 2my - 2ny + 2mn$.

6. $x^2 + w^2 + 9z^2 + 25 + 2xw - 6xz + 10x - 6wz + 10w - 30z$.

7. $a - x + 3y$.

8. $2m + 5n - 6p$.

9. $a - 2b - 3c - 7$.

Exercise 142

1. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $8 - 12a + 6a^2 - a^3$.
3. $8x^3 + 12x^2y + 6xy^2 + y^3$.
4. $27m^3 - 54m^2n + 36mn^2 - 8n^3$.
5. $8a^3b^3 + 12a^2b^2c + 6abc^2 + c^3$.
6. $x^6 - 9x^4y^2 + 27x^2y^4 - 27y^6$.
7. $(3 - x)^3$.
8. $(4 - 5a)^3$.
9. $(2a - 3b)^3$.
10. $(a^2 - 3x)^3$.

Exercise 143

1. $x^3 + y^3$.
2. $x^3 - y^3$.
3. $m^3 + 8$.
4. $8a^3 - b^3$.
5. $x^6 + y^6$.
6. $8a^6 - 27b^3$.
7. $(m + n)(m^2 - mn + n^2)$.
8. $(m - n)(m^2 + mn + n^2)$.
9. $(x + 1)(x^2 - x + 1)$.
10. $(x - 1)(x^2 + x + 1)$.
11. $(a + 2)(a^2 - 2a + 4)$.
12. $(a + 2b)(a^2 - 2ab + 4b^2)$.
13. $(3m - 1)(9m^2 + 3m + 1)$.
14. $(2n + 3r)(4n^2 - 6nr + 9r^2)$.
15. $(mn + y)(m^2n^2 - mny + y^2)$.
16. $(5 + y)(25 - 5y + y^2)$.
17. $(ab - xy)(a^2b^2 + abxy + x^2y^2)$.
18. $(c + 7)(c^2 - 7c + 49)$.

19. $(a^2+1)(a^4-a^2+1)$.
20. $(5-m^2)(25+5m^2+m^4)$.
21. $(2m^2-3n^2)(4m^4+6m^2n^2+9n^4)$.
22. $(4a^3-x)(16a^6+4a^3x+x^2)$.
23. $(x^4+1)(x^8-x^4+1)$.
24. $(x^4+4)(x^8-4x^4+16)$.
25. $(x^3-y^2)(x^6+x^3y^2+y^4)$.
26. $[x-y-2][(x-y)^2+2(x-y)+4]$.

Exercise 144

1. $x^4+x^2y^2+y^4$.
2. $n^4+9m^2n^2+81m^4$.
3. $16a^4+36a^2b^2+81b^4$.
4. $x^8+x^4y^4+y^8$.
5. $256a^4+16a^2+1$.
7. $(x^2-x+1)(x^2+x+1)$.
8. $(1-a+a^2)(1+a+a^2)$.
9. $(x^4-x^2y^2+y^4)(x^2+xy+y^2)(x^2-xy+y^2)$.
10. $(a^2-a+2)(a^2+a+2)$.
11. $(a^2-3a+3)(a^2+3a+3)$.
12. $(a^2-3ab+5b^2)(a^2+3ab+5b^2)$.
13. $(x^2-3x+1)(x^2+3x+1)$.
14. $(2a^2-ab+2b^2)(2a^2+ab+2b^2)$.
15. $(x^2-2x+2)(x^2+2x+2)$.
16. $(2x^2-2x+1)(2x^2+2x+1)$.
17. $(8-4x+x^2)(8+4x+x^2)$.
18. $(8x^2-4x+1)(8x^2+4x+1)$.
19. $(x^2-x+\frac{1}{2})(x^2+x+\frac{1}{2})$.
20. $(x^2-x-1)(x^2+x-1)$.
21. $(x^2-3x-1)(x^2+3x-1)$.
22. $(x^2-4x-1)(x^2+4x-1)$.
24. $(x^2-4x+1)(x^2+4x+1)$.
25. $(x^2-5x+1)(x^2+5x+1)$.
26. $(x^2-6x+1)(x^2+6x+1)$.

Exercise 145

1. $(x-1)(x^2-6x-6)$. 4. $(a-2)(a^2+2a-16)$.
2. $(x+1)(x-3)(x+2)$. 5. $(x-1)^2(x-2)$.
3. $(a-1)(a^2-9a-9)$. 6. $(x-2)(x^2-4x-3)$.
7. $(x-2)(x+4)(x+1)$.
8. $(a+3)(a^2+9a-15)$.
9. $(m-2)(m^3-m^2-2m+1)$.
10. $(a-2)(a^2-3a-2)$. 11. $(a-3)(a^3-a-2)$.
12. $(m-2)(m^4+2m^3+4m^2+5m+34)$.
13. $(m-2n)(m^2-mn+2n^2)$.
14. $(a-n)(a-3n)(a+2n)$.
15. $(a-3y)(a^2-2ay-6y^2)$.

Exercise 146

2. $x^4-x^3y+x^2y^2-xy^3+y^4$.
3. $x^6+x^5y+x^4y^2+x^3y^3+x^2y^4+xy^5+y^6$.
4. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.
5. $(x-2)(x^4+2x^3+4x^2+8x+16)$.
6. $(x+3)(x^4-3x^3+9x^2-27x+81)$.
7. $(a^2+b^2)(a^4-a^2b^2+b^4)$.
8. $(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.
9. $(a^2+b^3)(a^4-a^2b^3+b^6)$.
10. $(a^2+5)(a^4-5a^2+25)$.

Exercise 147

1. $(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$.
2. $(x^2+4)(x^4-4x^2+16)$.
3. $(x^3+4)(x^6-4x^3+16)$.
4. $(x^4+y^4)(x^8-x^4y^4+y^8)$.
5. $(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$
$(x^2+y^2)(x^4-x^2y^2+y^4)$.
6. $(a-y)(a+y)^3$.

7. $(xy + 16)(xy + 1)$. 8. $(y - x - 1)(y + x + 1)$.
9. $(xy - 5 - 3z)(xy - 5 + 3z)$.
10. $(x^2 - 2x - 1)(x^2 + 2x - 1)(x^4 + 6x^2 + 1)$.
11. $(x + y)(x - y + 1)$. 12. $(x + 1)(x^2 - 2x + 5)$.
13. $(x^2 - 2x + 3)(x^2 + 2x + 3)$.
14. $(x^2 - 3x + 3)(x^2 + 3x + 3)$.
15. $(x^2 + 5y)(x^4 - 5x^2y + 25y^2)$.
16. $(x - 5)(x^2 - 10x - 50)$. 17. $(5x^2 - 6)(2x^2 - 7)$.
18. $(a^2 - 4b)(a^4 + 4a^2b + 16b^2)$.
19. $3(5ab - 6cd)(5ab + 6cd)$.
20. $(x - y)(x^2 + xy + y^2 + 1)$. 21. $(x + 1)(x^2 + 1)$.
22. $(x^4 - 2x^2 + 2)(x^4 + 2x^2 + 2)$.
23. $(2x - 1)(x + 2)$. 24. $(a + 2)(a^2 - 3a + 1)$.
25. $(6r^2 - 3r - 1)(6r^2 + 3r - 1)$.
26. $(5m - 2)(25m^2 - 20m + 1)$.
27. $(5x - 6)(2x + 3)$.
28. $(x + 1)(abx^2 - abx + ab + 1)$.
29. $(m - n)(m + n + m^2 + mn + n^2)$.
30. $(a - b)(a + b)(a^2 + b^2 - 1)$.
31. $(y + 1)(aby^2 - aby + ab + 1)$.
32. $7a(ax + 3)(ax + 4)$.
33. $(m - n - 12)(m - n + 3)$.
34. $(a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6)$.
35. $(a + 2)^2(2a - 1)$. 36. $(y + 1)(y^3 - 3)$.
37. $(x - 1)(x + 1)(3x^4 + 11x^2 + 3)$.
38. $[a^3 - b^3 + 2][(a^3 + 1)^2 + (a^3 + 1)(b^3 - 1) + (b^3 - 1)^2]$. 39. $(x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6)$.

Exercise 148

2. $(1 - x)(2 - x)(x - 3)$,
$-(1 - x)(x - 2)(x - 3)$, etc.

3. $\frac{-x}{a - 2b}$, $-\frac{-x}{2b - a}$, etc.

4. $\frac{b-a}{d-c}, -\frac{a-b}{d-c}, -\frac{b-a}{c-d}$

5. $\frac{b-c}{(b-a)(c-a)}$ $\frac{c-b}{(b-a)(a-c)}$ $\frac{c-b}{(a-b)(c-a)}$.

6. $\frac{a-b}{(c-a)(c-b)(a-d)}, \frac{a-b}{(c-a)(b-c)(d-a)},$
$\frac{a-b}{(a-c)(c-b)(d-a)}, \frac{b-a}{(c-a)(b-c)(a-d)},$
$\frac{b-a}{(a-c)(c-b)(a-d)}, \frac{b-a}{(a-c)(b-c)(d-a)}$
$\frac{b-a}{(c-a)(c-b)(d-a)}.$

Exercise 149

1. $\frac{1}{x^2+xy+y^2}.$

4. $\frac{x^2+xy+y^2}{x^4+x^3y+x^2y^2+xy^3+y^4}.$

2. $\frac{x+2}{x^2+2x+4}.$

5. $\frac{4x^2-2x+1}{(x+2)(x-2)(2x-1)}.$

3. $\frac{a+1}{a^2-a+1}.$

6. $\frac{(x-y)(x+y)}{x^4-x^2y^2+y^4}.$

7. $\frac{(x-2)(x+1)}{x^3+2x^2+4x-2}.$

8. $\frac{b-c-d}{d-c-b}.$

9. $\frac{x^4-x^2y^2+y^4}{x^8-x^6y^2+x^4y^4-x^2y^6+y^8}.$

10. $\frac{a-2b+2c}{a^2-2a(b-c)+4(b-c)^2}.$

Exercise 150

1. $\dfrac{2x^3}{x^2+xy+y^2}$.
3. $\dfrac{x^2y^2}{x^2+xy+y^2}$.
2. $\dfrac{x^4-5x^2+6}{x^2+x-2}$.
4. $\dfrac{x^5+3}{x+1}$.
5. $a^4-a^3b+a^2b^2-ab^3+b^4-\dfrac{2b^5}{a+b}$.
6. $x-2-\dfrac{x+7}{x^2-x-1}$.
7. $x^3-x^2+1-\dfrac{x+1}{x^2+x+1}$.
8. $a^2-ab+b^2+\dfrac{3}{a^2+ab+b^2}$.
9 $a^4+a^3c+a^2c^2+ac^3+c^4+\dfrac{2c^5}{a-c}$.

Exercise 151

1. $\dfrac{3x^2-3x-17}{(x-2)(x-3)(x-4)}$.
3. $\dfrac{-x-6}{(x+1)(x-1)}$.
4. $\dfrac{15}{(a-1)(a-4)(c-2)}$.
5. 0.
6. 0.
7. $\dfrac{2(a+1)}{a^2+a+1}$.
8. $\dfrac{-2a^2+6a+25}{(a+3)(a-3)}$.
9. 1.

Exercise 152

2. 1.
3. $\dfrac{y(x^2+3xy+y^2)}{(x+y)^2}$.
4. $(a+b)(a-b)$.
6. $\dfrac{4b^2}{(a+b)(a-b)}$.
7. $\dfrac{2x^2-a^2}{a^2}$.
8. $\dfrac{(a-x+b)^2}{2(a+b+x)(b+x-a)}$.
9. $-\dfrac{1}{a}$.
10. $\dfrac{x^3}{(x+y)^2}$.

Exercise 153

1. $x=2$.
2. $x=7$.
3. $x=11$.
4. $x=\frac{4}{5}$.
5. $x=2$.
6. $x=2$.
7. $x=1$.
8. $x=8$.
9. $x=1$.
10. $x=3$.
13. $x=5$.
14. $x=3$.
15. $x=-\frac{7}{8}$.
16. $x=3$.
17. $x=\frac{3}{8}$.
18. $x=6$.
19. $x=19$.
20. $x=23$.
22. $x=1$.
23. $x=9a$.
24. $x=\dfrac{4a+3b}{3a+4b}$.
25. $x=\dfrac{m^2}{n}$.
26. $x=\dfrac{9-2m^2}{2m}$.
27. $x=a^2+b^2$.
28. $x=\dfrac{a^2-b^2}{ab}$.
29. $x=\dfrac{2a-3b}{3a-2b}$.

Exercise 154

1. y^2-y+1.
2. k^3-3k^2+3k-1.
3. $-19,\ -27,\ -54$.
4. $0,\ -1,\ -\frac{2}{3}$.
5. $-\dfrac{51}{64},\ -\dfrac{51}{64}$.
6. $11,\ 10,\ -3,\ 11\frac{5}{8}$.
7. $0;\ a+2$.
8. $(x-3)(x-2)(x-1)$.
9. $a^2-4a+4,\ a^2,\ (a+y)^2-2(a+y)+1$.

Exercise 155

1. $x=7,\ y=4$.
2. $x=3,\ y=2$.
3. $a=5,\ b=-3$.
4. $m=\dfrac{3}{2},\ n=-\dfrac{1}{2}$.
5. $x=\dfrac{3}{2}\ \ y=\dfrac{1}{2}$.
6. $x=2,\ y=3$.
7. $x=-5,\ y=-3$.
8. $x=1,\ y=-\frac{2}{3}$.
9. $R=4,\ r=-2$.
10. $a=\frac{1}{2},\ b=-\frac{1}{4}$.
11. $m=\frac{1}{5},\ n=-\frac{1}{4}$.
12. $x=6,\ y=-9$.
13. $x=\frac{1}{2},\ y=7$.
14. $x=7,\ y=3$.
15. $x=6,\ y=12$.
16. $x=\dfrac{5}{2},\ y=\dfrac{7}{10}$.

17. $x = 2,\ y = 1.$

18. Their graphs are the same.

19. Their graphs are parallel.

20. $x = -1,\ y = -2.$

21. $x = \frac{1}{2},\ y = -\frac{3}{2}.$

22. $x = \frac{2}{3},\ y = \frac{3}{2}.$

23. $x = 12,\ y = 25.$

24. $x = 5,\ y = 1.$

25. $x = 15,\ y = 25.$

26. $x = 4500,\ y = 1000.$

27. $x = a + b,\ y = a - b.$

28. $x = b,\ y = c.$

29. $x = mn,\ y = -m - n.$

30. $x = -1,\ y = 0.$

31. $x = a + b,\ y = a - b.$

32. $x = 2,\ y = -3.$

33. $x = \frac{1}{6},\ y = \frac{1}{4}.$

34. $x = \frac{2}{m + n},\ y = \frac{2}{n - m}.$

35. $x = \frac{1}{b},\ y = \frac{1}{a}.$

Exercise 156

2. $-1.$

3. $-10.$

4. $-8.$

5. $ad - bc.$

6. $14a^2 + 12b^2.$

7. $a^2.$

8. $-2a^2b^2.$

9. $-x^2 + 2y^2.$

10. $a^2 + 9ab - 5b^2.$

11. $x = -2,\ y = 2.$

12. $x = \frac{3}{2},\ y = \frac{1}{2}.$

13. $x = -2,\ y = -2.$

14. $x = \frac{3}{2},\ y = -\frac{1}{2}.$

15. $x = 2,\ y = 2.$

16. $x = 1,\ y = 2.$

17. $x = -1,\ y = -1.$

18. $x = 12,\ y = 8.$

19. $x = 20,\ y = 15.$

20. $x = 3\frac{1}{2},\ y = 4\frac{1}{2}.$

Exercise 157

1. $x = 1,\ y = -1,\ z = 3.$

2. $x = 60,\ y = 60,\ z = 20.$

3. $x = 1,\ y = 2,\ z = 3.$

4. $x = 3\frac{1}{2},\ y = 1\frac{1}{2},\ z = 5\frac{1}{2}.$

5. $a = 8\frac{5}{7},\ b = 3\frac{1}{7},\ c = 3.$

6. $x = 1,\ y = 5,\ w = 4,\ z = 2.$

7. $w = 4,\ x = 7,\ y = 6,\ z = 5.$

8. $x = 1,\ y = 2,\ z = 3,\ w = 4.$

9. $x = \frac{1}{6},\ y = \frac{1}{8},\ z = \frac{1}{10}.$

10. $x = 1,\ y = \frac{1}{2},\ z = \frac{1}{2}.$

Exercise 158

2. 31. **3.** 12. **4.** 0. **5.** $44a-3b$.
6. $a^3+b^3-a^2b-ab^2$. **10.** $x=-2,\ y=0,\ z=-3$.
7. $x=1,\ y=2,\ z=3$. **11.** $x=3,\ y=-5,\ z=7$.
8. $x=3,\ y=-2,\ z=5$. **12.** $x=\frac{1}{2},\ y=\frac{1}{3}\ \ z=-\frac{5}{2}$.
9. $a=2,\ b=-3,\ c=5$.

Exercise 159

1. 15 da. **2.** 15 hr., 30 hr. **3.** 6 hr., 8 hr., 12 hr.
4. 12 hr.; 10 hr., 15hr.; empty in 60 hr. **5.** $\frac{12}{19}$ lb.
6. 8 liters, 12 liters. **7.** 10 qt. **8.** 57. **9.** 257.
10. 5 hr. **11.** 40 mi. **12.** 15 min. **13.** 4 ft.
14. 4 ft. from fulcrum. **15.** $3\frac{1}{3}$ qt. **16.** 12 hr., 9 hr.
17. 1920 lb., 80 lb. **18.** 36. **19.** 60 lb.
20. \$9,200, \$2,800. **21.** 88%, 70%. **22.** \$3,600, \$7,200.
23. 750 lb. **25.** 10 hr., 12 hr., 20 hr.
24. $90\frac{10}{11}$ lb. **26.** 54 lb., 36 lb.
27. 325.
28. \$10,080, \$8,640. **29.** 30 min. **30.** 52. **31.** 3.
32. 60¢, 45¢. **33.** 25 lb., 35 lb. **34.** 48. **35.** 9 oz., 6 oz.

Exercise 160

2. 9. **3.** 1. **4.** 6. **5.** 6. **6.** 13; 37 cans, 38 cans.

Exercise 161

1. a^{15}. **2.** a^{2m}. **3.** b^{5x}. **4.** 10,000,000. **5.** m^{ab}. **6.** a^{2mx}. **7.** a^{mnx}.
8. a^{m^2}. **9.** x^{6a^2}. **10.** $a^{m^2+n}b^x$. **11.** $a^{n+2}b^{n+2}$. **12.** $3^5\cdot 2^5$. **13.** $2^4\cdot 3^6\cdot 5^{10}$. **14.** $a^{2nx}b^{2nx}$.
15. ab. **16.** $3bc$. **17.** $5^3\cdot 3^2$. **18.** $8a^{n-2}b^{n-3}$. **19.** abc. **20.** x^5yz^2. **21.** $x^{3a+3y}b^{7y-3a}$.

Exercise 162

1. a^5.
2. a^2c^3.
3. $32a^5$.
4. 1.
5. $(ax - by)^5$.
6. 29.
7. 10.
8. $16\frac{1}{4}$.

Exercise 163

2. $\frac{a}{3b}$.
3. $\frac{bd}{ac}$.
4. $\frac{2c}{ab}$.
5. $\frac{a^2c^3}{6b^3}$.
6. $\frac{m}{x(a-b)}$.
8. $\frac{12a^2c^3}{(x-y)^3}$.
10. $\frac{y+x}{y-x}$.
11. $\frac{xy}{x+y}$.
12. $\frac{xy(x+y)}{x^2+xy+y^2}$.
13. $\frac{xy}{2x+y}$.
14. $\frac{a^2b^2}{9a^2+3ab+b^2}$.
16. $\frac{225}{49}$.
17. $\frac{1}{12}$.
18. $\frac{1}{2}$.
19. 4.
21. $3 \cdot 2^{-1}xym^{-2}$.
23. $3xy(x-y)^{-1}$.
24. $2c(b-c)^{-2}$.
25. $3(x-y)^{-2}$.
26. $(a-b)^2(m^2+mn+n^2)^{-1}$.

Exercise 164

1. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2}$.
2. $\frac{1}{x^3} - \frac{3}{x^2y} + \frac{3}{xy^2} - \frac{1}{y^3}$.
3. $\frac{1}{a^3} - \frac{3}{a^2\text{b}} + \frac{3}{ab^2} - \frac{1}{b^3}$.
4. $\frac{1}{x^3} - \frac{3}{x^2} - \frac{4}{x} + 12$.
5. $\frac{1}{x} + 2$.
6. $\frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2}$.
7. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2}$.
8. $\frac{1}{a^4} - \frac{1}{a^3b} + \frac{1}{a^2b^2} - \frac{1}{ab^3} + \frac{1}{b^4}$.
9. $\frac{1}{x^2} - \frac{2}{x} + 1$.
10. $\frac{1}{x^2} + \frac{1}{x} - 1$.
11. $\frac{1}{x^4} - \frac{4}{x^3} + \frac{6}{x^2} - \frac{4}{x} + 1$.
12. $\frac{y^6}{x^3} - 3y^3 + 3x^3 - \frac{x^6}{y^3}$.
13. $\frac{8}{a^2} - \frac{18}{a} - 47 - 15a$.
14. $\frac{b^2}{a^2} - 2 + \frac{a^2}{b^2}$.

Exercise 165

1. $2^{\frac{1}{2}}$.
2. $3^{\frac{1}{3}}$.
3. $2a^{\frac{1}{2}}$.
4. $3a^{\frac{2}{3}}$.
6. $a^{\frac{5}{3}}$.
7. $3(a+b)^{\frac{5}{3}}$.
10. $3xy^{\frac{1}{3}}$.
12. $a^{\frac{5}{3}}b^{\frac{4}{3}}$.
13. $3(a+b)^{\frac{3}{2}}$.
14. $2x^{\frac{5}{3}}y^{\frac{4}{3}}$.
15. $2x^{\frac{1}{3}}y^{\frac{1}{3}}$.
17. $3^{\frac{3}{2}}(a+b)^{\frac{3}{2}}$.
18. $(m-n)^{\frac{4}{3}}$.
20. $\sqrt[3]{b^2}$.
21. $\sqrt[4]{c^3}$.
22. $\sqrt[n]{x^m}$.
24. $\sqrt[3]{9a^2b^2}$.
25. $3ab\sqrt[3]{3ab}$.
27. $z\sqrt[3]{xy^2z}$.
28. $\sqrt[b]{x^ay^a}$.
31. $\sqrt{\frac{a}{2x}}$.
32. $\frac{1}{\sqrt{a+b}}$.
33. $\frac{3}{\sqrt[3]{(x+y)^2}}$.
34. $\sqrt[3]{\frac{a}{4(b-c)}}$.
35. $\frac{3\sqrt[4]{a^2c^3}}{\sqrt{b}}$.
37. $2ax^{-\frac{1}{2}}y^{\frac{1}{3}}$.
38. $3a^{\frac{1}{2}}b(a+b)^{-\frac{1}{2}}$.
39. $2\cdot 3^{-1}ab^{-1}(x-y)^{-3}$.
40. $abc(a+b)^{-\frac{2}{3}}$.
41. $x^2y^2(x+y)^{-1}$.

Exercise 166

1. 3.
3. 8.
4. 27.
5. 32.
7. $\frac{1}{32}$.
8. $\frac{1}{243}$.
9. $\frac{1}{16}$.
10. 2.
11. 4.
12. 32.
13. 3.
14. $\frac{1}{25}$.
15. 16.
16. 32.
17. 5.
18. 5.
19. $\frac{1}{12}$.
20. 4.
21. $\frac{1}{16}$.
22. 1000.
23. 1.5.
24. $\frac{5}{6}$.
25. 4.
29. $3^{-\frac{1}{3}}$.
30. 4.
31. 16.
32. 27.
33. $\frac{1}{4^{\frac{2}{3}}}$.
34. 4.
35. $\frac{1}{4}$.
37. $\frac{3}{2}$.
38. $576\sqrt{2}$.

Exercise 167

1. $\sqrt[3]{6}$.
2. $\sqrt{5}$.
3. $\sqrt{6}$.
4. $\sqrt[3]{5}$.
5. $2\sqrt{3}$.
6. $2\sqrt[4]{3}$.
7. $3\sqrt[3]{2}$.
8. $5\sqrt[3]{2}$.
9. $2a\sqrt[4]{a}$.
10. $3a\sqrt[3]{3ab^2}$.
11. $2xy\sqrt[3]{7y}$.
12. $7\sqrt[3]{2}$.
13. $2\sqrt[4]{8}$.
14. $8\sqrt[3]{2}$.
15. $3\sqrt[5]{3}$.
17. $2\sqrt[6]{2}$.
18. $2\sqrt[3]{18}$.
19. $\frac{1}{5}\sqrt{10}$.
20. $\frac{1}{b}\sqrt{ab}$.
22. $\frac{2}{3a}\sqrt[3]{6a^2}$.
23. $\frac{1}{2}\sqrt[3]{4}$.
24. $\frac{1}{2}\sqrt[4]{8}$.
25. $\frac{1}{2}\sqrt[5]{16}$.
26. $\frac{1}{5a}\sqrt[3]{10a^2}$.
27. $\frac{3}{a}\sqrt[3]{4a}$.
28. $\sqrt[4]{2ab^2}$.
29. $\frac{x}{3y}\sqrt[3]{4xy}$.
30. $\frac{1}{a+b}\sqrt{a+b}$.
31. $\frac{1}{a+b}\sqrt[3]{a(a+b)}$.
32. $\frac{2}{m+n}\sqrt[3]{a^2(m+n)^2}$.
33. $\sqrt{x^2-y^2}$.
34. $\sqrt[3]{a^2-b^2}$.
36. $\frac{s}{6\pi}\sqrt{s\pi}$.
37. $\sqrt[3]{36v^2\pi}$.

Exercise 168

1. .815.	5. .659	9. 22.624.	13. 26.46.
2. .737.	6. .268.	10. 7.56.	14. 7.935.
3. .894.	7. 1.764.	11. 11.312.	15. 12.6.
4. .909.	8. .437.	12. 17.888.	16. .318.

Exercise 169

2. $4\sqrt{2}$.
3. $6\sqrt{3}$.
5. $3\sqrt{6}$.
6. $8\sqrt[3]{3}$.
7. $2\sqrt{6}$.
8. $-17\sqrt{3}$.
9. $\sqrt[3]{6}$.
10. $4\sqrt[3]{4}$.
11. $-16\frac{1}{2}\sqrt[3]{5}$.
12. $6a\sqrt{a}$.

14. $\dfrac{a+b-2ab}{ab}\sqrt{ab}.$ 15. $(a+b)^2\sqrt{a-b}.$

Exercise 170

1. $\sqrt[3]{4}.$
2. $\sqrt[5]{6} > \sqrt{2}.$
3. $\sqrt{7} > \sqrt[3]{18}.$
4. $\sqrt[6]{19} > \sqrt[4]{7}.$
5. $\sqrt[3]{20} > \sqrt[4]{50} > \sqrt{7}.$
6. $\sqrt[4]{5} > \sqrt[6]{10} > \sqrt[3]{3} > \sqrt{2}.$
7. $3\sqrt[3]{2} > \sqrt[6]{2880} > 2\sqrt{3}.$
8. $2\sqrt{3} > \sqrt[6]{1700} > 2\sqrt[3]{5}.$
9. $4\sqrt[4]{10} > 5\sqrt{2} > 2\sqrt[3]{6}.$

Exercise 171

2. $x\sqrt[10]{x}.$
3. $\sqrt[12]{a^5}.$
4. $3\sqrt[6]{a}.$
5. $2a\sqrt[4]{a}.$
6. $2\sqrt[4]{a}.$
7. $9a\sqrt[3]{b^2}.$
8. $7^7\cdot 4x^2y^3\sqrt[3]{2x}\sqrt{y}.$
9. $x-2\sqrt{xy}+y.$
10. $x\sqrt{x}+3x\sqrt{y}+3y\sqrt{x}+y\sqrt{y}.$
11. $x-2\sqrt{x}\sqrt[3]{y}+\sqrt[3]{y^2}.$
12. $x\sqrt{x}-3x\sqrt[3]{y}+3\sqrt{x}\sqrt[3]{y^2}-y.$
13. $4a^{\frac{2}{3}}-12a^{\frac{1}{3}}b^{\frac{1}{2}}+9b.$
14. $a-3a^{\frac{2}{3}}b^{\frac{1}{2}}+3a^{\frac{1}{3}}b-b^{\frac{3}{2}}.$
15. $x\sqrt{x}+y\sqrt{y}.$
16. $x^3-y^3.$
17. $\sqrt{x}+\sqrt{y}.$
18. $\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}.$
19. $\sqrt{x}+\sqrt{y}.$

Exercise 172

4. $\sqrt[3]{25}.$ 5. $\sqrt[4]{2}.$ 6. $\sqrt[4]{8}.$ 7. $\sqrt[5]{a}.$ 8. $\sqrt[3]{ab^2}.$

9. $\sqrt[3]{4a^2b}.$ 10. $\sqrt[4]{3a^2b^3}.$ 12. $\sqrt{7}+\sqrt{5}.$

13. $2\sqrt{3}+3\sqrt{2}.$ 14. $\sqrt{a}-\sqrt{b}.$

16. $(\sqrt{3}+\sqrt{2}+1)\sqrt{2}.$

17. $(\sqrt{a}+\sqrt{b}-c)(a-b-c^2-2c\sqrt{b}).$

Exercise 173

2. $\sqrt{2}.$
3. $\sqrt[3]{4}.$
4. $\sqrt[3]{3}.$
5. $a\sqrt{a}.$
6. $\sqrt[3]{2a^2}.$
7. $\sqrt[4]{2a^2b}.$
8. $2\sqrt[3]{36}.$
9. $\frac{\sqrt[3]{7}}{2}.$
10. $\sqrt{2}.$
11. $\frac{3\sqrt[3]{25}}{2}.$
12. $3(\sqrt{2}+1).$
13. $2(\sqrt{3}+1).$
14. $\sqrt{7}-\sqrt{2}.$
15. $\frac{3-6\sqrt{2}+\sqrt{6}-4\sqrt{3}}{3}.$
16. $\frac{5\sqrt{6}-6}{19}.$
17. $\sqrt{a}+\sqrt{b}.$
18. $\frac{ab-(b+a)\sqrt{ab}}{b^2+ab+a^2}.$
19. $\frac{7(\sqrt{6}+\sqrt{2})}{2}.$
20. $\sqrt{14}+3.$
21. $12-5\sqrt{6}.$
22. $\frac{(\sqrt{3}+\sqrt{2}-\sqrt{5})\sqrt{6}}{2}.$
23. $(1+\sqrt{2}-\sqrt{3})\sqrt{2}.$
24. $\frac{(\sqrt{a}+\sqrt{b}-\sqrt{a+b})\sqrt{ab}}{2}.$

Exercise 174

2. $x=3.$
4. $x=3.$
5. $x=\frac{3}{2}.$
6. $x=\frac{3}{4}.$
7. $x=\frac{2}{3}.$
8. $x=\frac{1}{2}.$
9. $x=-\frac{1}{2}.$
10. $x=\frac{4}{5}.$
11. $x=-\frac{4}{3}.$
12. $x=\frac{2}{3}.$
13. $x=-3.$
14. $x=4.$
15. $x=3.$
17. $x=-2.$
18. $x=-\frac{4}{3}.$
19. $x=4.$
20. $x=729.$
21. $x=\frac{1}{64}.$
23. $x=\frac{16}{81}.$
24. $x=.01.$
25. $a=8.$
26. $a=\frac{27}{125}.$
27. $a=\frac{3}{10}.$

Exercise 175

1. $\sqrt{5}-\sqrt{3}.$ **5.** $2\sqrt{2}-\sqrt{3}.$ **9.** $\sqrt{2a}+\sqrt{3b}.$

2. $\sqrt{7}-1.$ **6.** $\sqrt{6}+\sqrt{3}.$ **10.** $1-\frac{1}{2}\sqrt{2}.$

3. $3-\sqrt{3}.$ **7.** $5-\sqrt{6}.$ **11.** $3-\frac{1}{3}\sqrt{3}.$

4. $\sqrt{6}-\sqrt{5}.$ **8.** $\sqrt{a}-\sqrt{b}.$ **12.** $a+\sqrt{b}.$

Exercise 176

3. $6i\sqrt{3}-3i.$ **4.** $-3ai.$ **5.** $13i\sqrt{2}+7i\sqrt{3}.$

6. $18i\sqrt{2}-5i\sqrt{3}.$ **8.** $(2ai+2i+3a^2i)\sqrt{a}.$

7. $27i\sqrt{6}-5i\sqrt{5}.$ **9.** $(3xi-7ai+6a^2x^2i)\sqrt{x}.$

10. $-9.$ **14.** $-42i\sqrt{abc}.$ **18.** $4a+9b.$

11. $-6\sqrt{6}.$ **15.** $-6a^2bi\sqrt{b}.$ **19.** $-7+4i\sqrt{15}.$

12. $27\sqrt{2}.$ **16.** $abcd.$ **20.** $7i\sqrt{2}-3\sqrt{3}.$

13. $6\sqrt{6}.$ **17.** 11 **21.** 154.

22. $-10-2\sqrt{6}+2\sqrt{10}+2\sqrt{15}.$

23. $-a-b-c-9-2\sqrt{ab}-2\sqrt{ac}-2\sqrt{bc}$
$+6\sqrt{a}+6\sqrt{b}+6\sqrt{c}.$

24. $-2i\sqrt{3}.$ **25.** $-2i\sqrt{a}.$ **26.** $\sqrt{3}+\sqrt{-2}.$

27. $\dfrac{(ai\sqrt{b}+bi\sqrt{a})}{b-a}.$ **28.** $-1+i\sqrt{35}.$

31. $3-\sqrt{-5}.$

32. $3-2\sqrt{-2}.$ **34.** 0. **35.** Yes.

Exercise 177

1. $x^2 - x - 2.$
2. $2a^2 - 3ab - 2b^2.$
3. $2a + 3b - 5c.$
4. $n^2 - n + \frac{1}{2}.$
5. $\frac{3a^2}{4} - \frac{a}{2} + \frac{1}{4}.$
6. $3a^{\frac{2}{3}} - 2a^{\frac{1}{3}} - 2.$
7. $3x^{-\frac{2}{3}} - 2x^{-\frac{1}{3}} - 3.$
8. $2n^{-1} - 3 + 2n.$
9. 225.
10. 373.
11. 302.3.
12. 4.27.
13. 8070.
14. 1636.
15. 6.245.
16. 1.414.
17. 2.646.
18. 5.413.
19. 18.388.
20. .943.
21. $6n^{-4} - 7n^{-3} - n^{-2} - 14n^{-1} - 8.$
22. $2a^{\frac{4}{3}} - 7a + 7a^{\frac{2}{3}} - 4a^{\frac{1}{3}} - 10.$
23. $2x^{\frac{3}{3}} - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 3.$
24. $\sqrt{3}.$
25. $6\sqrt{5}.$
26. $\frac{1}{2} - \sqrt{3}.$
27. $\frac{3\sqrt{5} + 3\sqrt{2} + 7 - 2\sqrt{10}}{3}.$
28. $\frac{1}{2}\sqrt{6} - 5.$
29. $3\frac{1}{3}\sqrt{5}.$
30. $2\sqrt{15}.$
31. $\frac{m^2n^2}{(m + n)^2}.$
32. $ab(a + b).$
33. 0.
34. 19.799.
35. .732.
36. $-4.271.$
37. $\sqrt{6} - \sqrt{3}.$
38. $\sqrt{m + n} + \sqrt{m - n}.$
39. $\sqrt{5} + \sqrt{-3}.$
40. $a - \sqrt{-b}.$
41. 6.
42. Yes.

Exercise 178

1. $x = 3.$
2. $x = 3.$
3. $x = -3.$
4. $x = -2.$
6. $x = 2.$
7. $x = -\frac{3}{2}.$
9. $x = -2.$
10. $x = -3.$
11. $x = \frac{3}{2}.$
12. $x = \frac{2}{3}.$
13. $x = -2.$
14. $x = \frac{3}{4}.$

Exercise 179

2. $\log 299 = 2.4757$.
3. $\log 425 = 2.6284$.
4. $\log 755 = 2.8779$.
5. $\log 333 = 2.5224$.
6. $\log 129 = 2.1106$.
7. $\log 999 = 2.9996$.
8. $\log 500 = 2.6990$.
9. $\log 852 = 2.9304$.
10. $\log 102 = 2.0086$.
11. $\log 771 = 2.8871$.
12. $\log 101 = 2.0043$.

Exercise 180

1. 2.
2. 1.
3. 0.
4. $9. \cdots - 10$.
5. $8. \cdots - 10$.
6. $6. \cdots - 10$.
7. 5.
8. 4.
9. 8.
10. 3.
11. 1.
12. $1. \cdots - 10$.

14. $\log 57700 = 4.7612$.
15. $\log 4590 = 3.6618$.
16. $\log .00577 = 7.7612 - 10$.
17. $\log 3580000 = 6.5539$.
18. $\log 93000000 = 7.9685$.
19. $\log 5280 = 3.7226$.
20. $\log .00004 = 5.6021 - 10$.
21. $\log .000000435 = 3.6385 - 10$.

Exercise 181

1. 16.6.
2. 200000.
3. 3070.
4. .0755.
5. .00443.
6. .755.
7. 2.
8. 99800000.

Exercise 182

1. $\log 44.45 = 1.6479$.
2. $\log 234.7 = 2.3705$.
3. $\log 125800 = 5.0997$.
4. $\log .002755 = 7.4401 - 10$.
5. $\log 3.141 = 0.4970$.
6. $\log 9990 = 3.9996$.
7. $\log 7926 = 3.8991$.
8. $\log 4.188 = 0.6220$.
9. $\log .3125 = 9.4949 - 10$.
10. $\log .0007224 = 6.8587 - 10$.
11. $\log 6666 = 3.8239$.
12. $\log 1728 = 3.2375$.
13. $\log 1.732 = 0.2385$.

Exercise 183

1. 357.7.
2. 4556.
3. 85.2.
4. 3.141.
5. .6666.
6. 1728.
7. 1.732.
8. 31.62.
9. .0007345.
10. 986.8.
11. .1258.
12. 5455.
13. .002983.
14. 3980000.

Exercise 184

1. 8760.
2. 7.465.
3. 500.
4. 6.042.
5. 1.442.
6. 2.646.
7. 331800.
8. .0006141.
9. .12.
10. 1.653.
11. 2.632.
12. .0000004768.
13. 2.987.
14. .1913.
15. 14.91.
16. 3.339.
17. 1195.
18. 1.005.
19. 5.311.
20. .2378.

Exercise 186

1. 500 seconds.
2. 46940000000000 miles.
3. 1760000000000000 miles.
4. $6,692.
5. $6,714.
6. $193.60.
7. 11.9 years.
8. 11.75 years.
9. 6.96 years.
10. $7,446.60.
11. $7,473.30.
12. 1963.5 square inches.
13. 6.77 inches.
14. 62.35 square inches.
15. 176.1 square inches.
16. 498.8 square inches.
17. 8.485 inches.
18. 7.445 inches.
19. 6.443 inches.
20. 1256.6 square inches.
21. 197000000 square miles.
22. $236 \cdot 10^{10}$ square miles.
23. 3.385 inches.
24. 7238 cubic inches.
25. 7.443 inches.
26. $2601 \cdot 10^{8}$ cubic miles.
27. $34 \cdot 10^{16}$ cubic miles.
28. $9198 \cdot 10^{9}$ pounds.
29. 130700.
30. 26400.
31. 194.4 square inches.
32. 7.615 acres.
33. 18.18 inches.
33. 18.23 inches.
34. 110.2 pounds.
35. 104.9 pounds.
36. 1527 cubic inches.
37. 1204 pounds.

Exercise 187

1. $x = 5,\ x = -5.$
2. $a = 3,\ a = 4.$
3. $x = 8,\ x = 3.$
4. $x = \frac{7}{2},\ x = -\frac{7}{2}.$
5. $a = -1,\ a = -\frac{8}{3}.$
6. $n = \frac{7}{5},\ n = -1.$
7. $x = 0,\ x = 9,\ x = -9.$
8. $a = \frac{2}{3},\ a = \frac{3}{2}.$
9. $n = \frac{5}{2},\ n = -\frac{4}{3}.$
10. $x = 0,\ x = 9,\ x = -6.$
11. $a = 0,\ a = \frac{2}{3},\ a = 1.$
12. $x = 2,\ x = -2,\ x = 1,\ x = -1.$
13. $x^2 - 9x + 20 = 0.$
14. $x^2 + 3x - 18 = 0.$
15. $x^2 + 9x + 14 = 0.$
16. $x^2 + 2x - 35 = 0.$
17. $x^2 - ax - bx + ab = 0.$
18. $x^2 + bx - cx - bc = 0.$
19. $x^2 - 2bx - 3cx + 6bc = 0.$
20. $x^2 + 3ax - 4bx - 12ab = 0.$
22. $2x^2 - 3x - 2 = 0.$
23. $3x^2 + 7x - 6 = 0.$
24. $6x^2 + 13x + 6 = 0.$
25. $3n^2 - 2nx - x^2 = 0.$
26. $6x^2 - 11nx + 3n^2 = 0.$
27. $2ax^2 - 4a^2x - x + 2a = 0.$
28. $x = \pm 5.$
29. $x = \pm \frac{3}{2}.$
30. $x = \pm \sqrt{7}$, or $x = \pm 2.646.$
31. $x = \pm 1.9365.$
32. $x = \pm \frac{2ab}{3}.$
33. $x = \pm \frac{2}{7}\sqrt{14a}.$
34. $a^2 + 8a + 16.$
35. $n^2 - 10n + 25.$
36. $9a^2 - 6a + 1.$
37. $x^2 + 12ax + 36a^2.$
38. $4n^2 - 12n + 9.$
39. $x^2 + x + \frac{1}{4}.$
40. $4a^2 - 2ab + \frac{1}{4}b^2.$
41. $9x^2 - 3x + \frac{1}{4}.$
42. $9a^2 - 3a + \frac{1}{4}.$
43. $\frac{4}{9}a^2 - \frac{16}{3}ay + 16y^2.$

Exercise 188

1. $n = 4,\ n = -10.$
2. $a = 10,\ a = 2.$
3. $x = 9,\ x = -2.$
4. $n = 5,\ n = -8.$

5. $x = -\frac{5}{3}$, $x = -1$.
6. $a = \frac{3}{5}$, $a = -2$.
7. $a = \frac{5}{2}$, $a = -\frac{2}{3}$.
8. $x = -1.586$, $x = -4.414$.
9. $a = 9.243$, $a = .757$.
10. $n = 1.189$, $n = -2.523$.
11. $a = 1.535$, $a = -.869$.
12. $x = 1$, $x = -\frac{17}{5}$.
13. $x = 1.193$, $x = -4.193$.
14. $x = 3$, $x = -\frac{1}{2}$.
15. $n = 1$, $n = -3$.
16. $x = \frac{8}{3}$, $x = -1$.

Exercise 189

1. $a = \frac{3}{2}$, $a = -3$.
2. $n = \frac{7}{4}$, $n = -1$.
3. $x = 2$, $x = -\frac{3}{2}$.
4. $x = \frac{2}{3}$, $x = -\frac{1}{2}$.
5. $n = \frac{7}{3}$ $n = -1$
6. $n = 2.430$, $n = -1.097$.
7. $a = 1$, $a = -4$.
8. $n = \frac{3}{5}$, $n = -1$.
9. $x = 1.386$, $x = -2.886$.
10. $x = .7015$, $x = -5.7015$.

Exercise 190

1. $x = 1$, $x = \frac{1}{2}$.
2. $x = 1$, $x = -\frac{1}{3}$.
3. $x = \frac{2}{3}$, $x = -1$.
4. $x = -2$, $x = -\frac{2}{3}$.
5. $a = \frac{8}{7}$ $a = -1$.
6. $n = \frac{5}{2}$, $n = -2$.
7. $x = \frac{1}{2}$, $x = -3$.
8. $a = \frac{6}{5}$, $a = -1$.
9. $a = .85$, $a = -2.35$.
10. $a = .907$, $a = -2.573$.
11. $a = -\frac{1}{4}$, $a = -1$.
12. $n = -2$, $n = -\frac{3}{2}$.
14. $x = \frac{-3 \pm \sqrt{-7}}{2}$.
15. $n = \frac{3 \pm \sqrt{-7}}{4}$.
16. $a = -.2764$, $a = -.7236$.

17. $x = \frac{-1 \pm \sqrt{-3}}{4}$.

19. $x = -1,\ x = \frac{1 \pm \sqrt{-3}}{2}$.

20. $x = 2,\ x = -1 \pm \sqrt{-3}$.

21. $x = -2,\ x = 1 \pm \sqrt{-3}$.

22. $x = \frac{-1 \pm \sqrt{-3}}{2},\ x = \frac{1 \pm \sqrt{-3}}{2}$.

23. $x = \frac{-3 \pm \sqrt{-3}}{2},\ x = \frac{3 \pm \sqrt{-3}}{2}$.

24. $x = 1,\ x = -1,\ x = \frac{-1 \pm \sqrt{-3}}{2},\ x = \frac{1 \pm \sqrt{-3}}{2}$.

25. $x = -\frac{3}{2n},\ x = \frac{1}{n}$.

26. $x = 2a,\ x = -\frac{a}{3}$.

27. $x = \frac{5a}{2},\ x = -2a$.

28. $x = \frac{a-b}{a},\ x = -1$.

29. $x = \frac{2a}{3b},\ x = \frac{3b}{2a}$.

30. $x = \frac{3b}{a},\ x = -\frac{2n}{c}$.

31. $x = \frac{5b}{2a},\ x = -\frac{2b}{3a}$.

32. $x = a,\ x = \frac{a+b}{2}$.

33. $x = n,\ x = -\frac{n}{7}$

34. $x = \frac{3a}{2},\ x = -\frac{3a}{2}$.

Exercise 191

1. Roots are real, rational, and unequal.
2. Roots are real, rational, and equal.
3. Roots are real, rational, and unequal.
4. Roots are real, irrational, and unequal.
5. Roots are imaginary.
6. Roots are imaginary.

7. $k=25$. **8.** $k=12$, $k=-12$. **9.** $m=\frac{1}{3}$.

10. $m=4$.

11. $x=\frac{3\pm\sqrt{5}}{2}$.

12. $x=\frac{1\pm\sqrt{-11}}{2}$.

13. m is greater than $\frac{1}{3}$.

14. $x=\frac{-1\pm\sqrt{-7}}{2}$.

15. $x=\frac{-1\pm\sqrt{-11}}{2}$.

Exercise 192

1. $x^2-2x-15=0$.
2. $x^2+11x+28=0$.
3. $3x^2-7x-6=0$.
4. $6x^2-25x+24=0$.
5. $x^2-6x+2=0$.
6. $x^2-7x+11=0$.
7. $9x^2+30x+22=0$.
8. $4x^2-16x+19=0$.
9. $4x^2-12x+11=0$.
10. $x^2+5x+7=0$.
11. Yes.
12. Yes.
13. $k=-20$, $x=-10$.
14. $n=2$, $x=1$.
15. $x=5$, $k=9$.
16. $m=13$, $n=40$.
17. $x=4$, $x=7$, $k=28$.
18. $x=3$, $x=6$, $k=18$.

Exercise 193

1. $x=5$.
2. $a=3$, $a=-1$.
3. $n=2$.
4. $x=7$, $x=15$.
5. $n=5$.
6. $m=5$.
7. $a=1$.
8. $x=7$.
9. $x=\frac{1}{2}$, $x=-3$.
10. $x=-4$, $x=-8$.
11. $x=-8$.
12. $x=2n$.

Exercise 194

1. $x=3$, $x=\frac{-3\pm3\sqrt{-3}}{2}$.

2. $x=3$, $x=\frac{-7\pm\sqrt{33}}{4}$.

3. $n=\frac{1\pm\sqrt{-3}}{2}$, $n=\frac{-1\pm\sqrt{-3}}{2}$.

4. $x=2$, $x=-2$, $x=\pm\sqrt{5}$.

5. $a=2$, $a=-2$, $a=-1\pm\sqrt{-3}$, $a=1\pm\sqrt{-3}$.

6. $x=25$.

7. $a=\frac{8}{27}$, $a=-8$.

8. $x=4096$.

9. $x=3$, $x=\frac{1}{2}$.

10. $x=\sqrt[3]{9}$, $x=1$.

11. $a=6$.

12. $a=1$, $a=-4$.

13. $x=\frac{3\pm\sqrt{5}}{2}$, $x=\frac{-3\pm\sqrt{5}}{2}$.

14. $x=\frac{\sqrt{-7}\pm 1}{2}$, $x=\frac{-\sqrt{-7}\pm 1}{2}$.

15. $x=\frac{3\pm\sqrt{-7}}{4}$, $x=\frac{-3\pm\sqrt{-7}}{4}$.

16. $x=\frac{1}{3}$, $x=\frac{1}{2}$.

17. $x=\frac{3}{2}$, $x=\frac{2}{3}$.

18. $x=1$, $x=2$, $x=\frac{1}{2}$.

19. $x=\frac{9}{19}$, $x=-9$.

20. $x=\frac{31}{11}$, $x=-19$.

Exercise 195

1. 9 and 10, -10 and -9.
2. 7 and 9, -9 and -7.
3. 5, 6, and 7, -7, -6, and -5.
4. 9 and 11.
5. 18 rods, 10 rods.
6. 4 boys.
7. 4 and 6.
8. 9 and 3, 16 and -4.
9. $\frac{-1\pm\sqrt{1+4k}}{2}$ and $k-\frac{-1\pm\sqrt{1+4k}}{2}$.

10. 5, 6, 7, and 8. **11.** 32 rods wide and 40 rods long.
12. 8 and 64, -9 and 81.
13. Side 10 inches, hypotenuse 26 inches.
14. 3 inches and 14 inches. **15.** 9 boys.

16. $n = 8$. $n = \frac{3 \pm \sqrt{9 + 8d}}{2}$.

17. 100°, 53°, and 27°. **18.** 81°, 108°, 75°, and 96°.
19. Altitude, 10.392 in. Area, 62.352 sq. in.
20. Its foot is 10 ft. from the wall and its top rests 17.32 ft. from the ground.
21. 10.76 in. and 8.784 in.
22. 3 seconds. **23.** 2.18 seconds.
24. Altitude, 6 in. Bases, 6 and 10 in.
25. 12 in. and 18 in.
26. 36. **27.** 20 sheep. **28.** 6 yards. **29.** 5 rods.

Exercise 198

1. When $x=5$, $y=4$; when $x=4$, $y=5$.
2. When $a=4$, $b=3$; when $a=3$, $b=4$.
3. When $m=7$, $n=2$.
4. When $x=6$, $y=-3$; when $x=-3$, $y=6$.
5. When $x=2$, $y=\frac{1}{2}$; when $x=\frac{1}{2}$, $y=2$.
6. When $x=2$, $y=1$; when $x=\frac{6}{5}$, $y=\frac{11}{5}$.
7. When $x=3$, $y=-1$; when $x=5$, $y=-2$.
8. When $x=2$, $y=3$; when $x=9$, $y=-4$.
9. When $x=3$, $y=-1$; when $x=\frac{3}{2}$, $y=-2$.
10. When $x=6$, $y=2$; when $x=0$, $y=-2$.
11. When $x=1$, $y=\frac{1}{2}$; when $x=\frac{1}{2}$, $y=1$.
12. When $x=\pm 9$, $y=\pm 3$.

13. When $x = \pm 3,\ y = \pm 2$; when $x = \pm 2,\ y = \pm 3$.

14. When $x = \pm 4,\ y = \pm 5$;

when $x = \pm 3\sqrt{3},\ y = \pm \sqrt{3}$.

15. When $x = \pm 1,\ y = \pm 2$;

when $x = \pm \frac{13}{110}\sqrt{55},\ y = \mp \frac{5}{22}\sqrt{55}$.

16. When $x = \pm 3,\ y = \pm 5$; when $x = \pm 8,\ y = \mp 5$.

17. When $x = \pm 7,\ y = \pm 3$;

when $x = \pm 9\sqrt{-1},\ y = \mp 11\sqrt{-1}$.

18. When $m = 3,\ n = 2$; when $m = 2,\ n = 3$;

when $m = -3 \pm \sqrt{3},\ n = -3 \mp \sqrt{3}$.

19. When $a = \pm 2,\ b = \pm 1$;

when $a = \frac{\pm\sqrt{11} \pm \sqrt{-1}}{2},\ b = \frac{\mp\sqrt{11} \pm \sqrt{-1}}{2}$.

20. See answers for No. 18.

21. When $x = 4,\ y = 3$; when $x = 3,\ y = 4$;

when $x = \frac{-9 \pm \sqrt{-31}}{2},\ y = \frac{-9 \mp \sqrt{-31}}{2}$.

22. When $x = \pm 3,\ y = -1$; when $x = -2,\ y = \pm \frac{3}{2}$.

23. When $a = 3,\ b = 1$; when $a = 1,\ b = -3$.

24. When $a = 2,\ b = 1$.

25. When $x = 4,\ y = 3$; when $x = 3,\ y = 4$.

26. When $x = \pm 2,\ y = \pm 1$; when $x = \pm 1,\ y = \pm 2$.

27. When $x = \pm 4,\ y = \pm 3$; when $x = \pm 3,\ y = \pm 4$.

28. When $m = a \pm 1,\ n = a \mp 1$.

29. When $r = 3,\ s = 2$; when $r = 2,\ s = 3$.

30. When $x = 2,\ y = 3$.

31. When $m = \pm 3,\ n = \pm 2$; when $m = \pm 2,\ n = \pm 3$.

32. When $a = \pm 1$, $b = \pm 2$; when $a = \pm 2$, $b = \pm 1$.

33. When $a = \pm 2$, $b = \pm 3$;

when $a = \pm \frac{6}{7}\sqrt{14}$, $b = \mp \frac{2}{7}\sqrt{14}$.

34. When $x = 2$, $y = \pm 5$; when $x = -4$, $y = \pm \sqrt{-11}$.

35. When $a = +5$, $b = +2$;

when $a = \pm \sqrt{6} - 1$, $b = \pm \sqrt{6} + 1$;
when $a = -2$, $b = -5$.

36. When $x = \pm 2$, $y = \pm 3$.

37. When $x = \pm 3$, $y = \pm 2$.

38. When $x = 2$, $y = 1$ and -2;
when $x = -3$, $y = 1$ and -2.

39. When $x = 2$, $y = -2$; when $x = -\frac{9}{2}$, $y = \frac{1}{6}$.

40. When $R = \frac{1}{2}$, $r = 1$;

when $R = \frac{-2 \mp \sqrt{3}}{2}$, $r = -2 \pm \sqrt{3}$.

41. When $x = 6$, $y = 3$; when $x = -3$, $y = -6$.

42. When $x = 15$, $y = 4$; when $x = -4$, $y = -15$.

43. When $x = \pm 4$, $y = \pm 2$.

44. When $x = \pm 20$, $y = \pm 10$.

45. When $x = \pm 3$, $y = \pm 1$;

when $x = \pm \frac{9}{2}\sqrt{-2}$, $y = \mp 4\sqrt{-2}$.

46. When $a = \pm 2$, $b = \pm 1$; when $a = \pm 1$, $b = \pm 2$.

47. When $x = \pm 2$, $y = \pm 1$; when $x = \pm 1$, $y = \pm 2$.

48. When $a = \pm 7$, $b = \pm 5$; when $a = \pm 5$, $b = \pm 7$.

49. When $a = 1$, $b = \frac{1}{2}$; when $a = \frac{1}{2}$, $b = 1$.

50. When $x = \pm b$, $y = \pm (a - b)$.

Exercise 199

1. 5 and -3.
3. 5 and 3, or -3 and -5.
2. 4 and 3, also $-\frac{7}{2}$ and $-\frac{9}{2}$.
4. 5 by 16 inches.
5. $\frac{a+\sqrt{2b-a^2}}{2}$, $\frac{a-\sqrt{2b-a^2}}{2}$.
6. 7 and 5, -5 and -7,

   or $\frac{\pm\sqrt{129}-3}{2}$ and $\frac{\pm\sqrt{129}+3}{2}$.
7. 4 and 3.
8. 3 with 1 or -2, -4 with 1 or -2.
9. 5 by 12 inches.
10. 36.
11. 5 and 3.
12. 7 and 11, or -1 and -13.
13. 24 in. and 10 in.
14. 9 boys each paying \$4.
15. 40 miles, 10 hours.
16. $5\frac{3}{7}$ in., $8\frac{4}{7}$ in.
17. 58.776 square inches.
18. 34.195 square inches.
19. 48 ft.
20. $\frac{\pm(b+1)\sqrt{ab}}{2b}$ $\frac{\pm(b-1)\sqrt{ab}}{2b}$.
21. Length of wire 50 ft., height of pole 48 ft.

Exercise 200

1. $\frac{2}{3}$.
2. $\frac{3}{5}$.
3. $\frac{7}{8}$.
4. $\frac{3b}{7a^3}$.
5. $\frac{x+y}{x-y}$.
6. $\frac{a+b}{a^2+ab+b^2}$.
7. $(x^2-y^2)(x+y)$.
8. $\frac{a}{c}=\frac{b}{d}$.
9. $\frac{x-y}{3y}=\frac{2x}{x+y}$.
10. $\frac{a^2}{b^2}=\frac{c^2}{d^2}$.

11. $\frac{x}{y}=\frac{a}{b}$.

12. $\frac{2x+5}{8}=\frac{2}{7-3x}$.

13. $\frac{2a}{a+b}=\frac{2c}{c+d}$.

14. $x=4$.

15. $x=\pm 16$.

16. $x=\pm 7\sqrt{2}$.

17. $x=\frac{1}{5}$.

18. $x=3,\ x=5$.

19. $x=\frac{1\pm\sqrt{5}}{2}$.

20. $x=0$.

21. ± 14.

22. $\pm 3\sqrt{3}$.

23. $\pm \frac{1}{4}\sqrt{2}$.

24. $\pm a^2b^3$.

25. $\pm(a+b)\sqrt{a-b}$.

26. $\pm(m-n)\sqrt{(m^2+mn+n^2)(m+n)(m^2+n^2)}$.

27. $\frac{x}{m}=\frac{n}{y}$.

28. $\frac{a-b}{x^2-x+1}=\frac{x^2+x+1}{a+b}$.

29. $x=5$.

30. $x=\pm 4\sqrt{3}$.

Exercise 201

1. 25. **2.** 45 miles. **3.** 270 miles, 360 miles.
4. 294 feet. **5.** 40 miles, 60 miles. **6.** \$12, \$18.
7. 12 inches, 14 inches. **8.** 24, 36, and 48 inches.
9. 125 feet. **10.** 6 inches, 9 inches.

11. Of a, $\frac{ab}{b+c}$ and $\frac{ac}{b+c}$; of b, $\frac{ab}{a+c}$ and $\frac{bc}{a+c}$; of c, $\frac{ac}{a+b}$ and $\frac{bc}{a+b}$.

12. Of 16, $\frac{144}{19}$ in. and $\frac{160}{19}$ in.; of 18, 8 in. and 10 in.; of 20, $\frac{160}{17}$ in. and $\frac{180}{17}$ in.

13. $8\frac{1}{3}$ feet. **14.** $6\frac{6}{7}$ feet and $9\frac{1}{7}$ feet.
15. 120 lbs. **16.** 13 feet. **17.** 840 miles.
18. 16.916, 20.712, and 25.373 inches. **19.** 100 feet.

20. 128 sq. in. **21.** 1:3. **22.** 20 inches.
23. 3 inches, 27 inches. **24.** 7 inches.
25. 3.056 inches, 4.944 inches. **26.** 4 and 16 inches.
27. 6 and 12 inches. **29.** \$800, \$1,000, and \$1,200.
28. 294 sq. in. **30.** $10\sqrt{2}$, $4\sqrt{2}$.

Exercise 202

2. $x = 4\frac{1}{4}$. **3.** Constant $\frac{4}{3}\pi$, variable r^3, $\frac{4}{3}\pi = 4.1888$.
4. 18. **6.** $d = kt^2$, $k = 16$.
5. $p = kav^2$. **7.** $4\sqrt{2}$ or 5.656 ft. $4\sqrt{5}$ or 8.944 ft.
8. 80 lbs. $31\frac{1}{4}$ lbs.
9. 1,656 miles. **11.** Yes. Safe load is $166\frac{2}{3}$ lbs.
10. 500 lbs. **12.** $1{,}041\frac{2}{3}$ lbs. 1,800 lbs.
13. 400 centimeters, 1,600 centimeters, 25 centimeters.
14. .707 seconds, .9 seconds, 3.162 seconds.

Exercise 204

2. $3 + 4 + 5 + 6 + 7$. **3.** $3 + 8 + 13 + 18$.
4. $6 + 6 + 6\frac{2}{3} + 7\frac{1}{2} + 8\frac{2}{5} + 9\frac{1}{3}$.
5. $-1 - \frac{2}{3} + 0 + \frac{4}{5} + \frac{5}{3}$.
6. $2 + 9 + 24 + 50 + 90 + 147$.
7. $\frac{5}{2} + \frac{14}{3} + \frac{27}{4} + \frac{44}{5} + \frac{65}{6} + \frac{90}{7}$.
9. $\frac{1}{5} + \frac{2}{7} + \frac{3}{9} + \frac{4}{11} + \frac{5}{13} \cdots$
10. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} \cdots$
11. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$

12. $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}\cdots$

13. $2+9+64+625+7776\cdots$

14. $\frac{1}{2}+\frac{2}{9}+\frac{3}{64}+\frac{4}{625}+\frac{5}{7776}\cdots$

Exercise 205

1. $l=31.$ **2.** $n=8.$ **3.** $a=11.$ **4.** $d=5.$ **5.** $l=19\frac{1}{2}.$ **6.** $n=7.$ **7.** $a=-1\frac{2}{3}.$ **8.** $d=-3.$ **9.** 15 feet. **10.** 304 feet. **11.** 5, 8, 11, 14. **12.** 47.

Exercise 206

2. 7, 13, 19, 25, 31, 37, 43.

3. -7, -1, 5, 11, 17, 23.

4. 2, $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, 4.

5. $a,\ \frac{3a+b}{4},\ \frac{a+b}{2},\ \frac{a+3b}{4},\ b.$

6. $\frac{a+b}{2}$; 21; $\frac{3x+2y}{2}$.

7. Every $1\frac{2}{3}$ inches.

8. 9 and 15.

Exercise 207

1. $d=3,\ s=100.$

2. $n=9,\ s=99.$

3. $n=7,\ d=-3.$

4. $l=26,\ s=51.$

5. $d=2\frac{1}{2},\ l=25.$

6. $n=6,\ l=20.$

7. $a=-5,\ s=-124.$

8. $a=\frac{2}{3},\ d=\frac{2}{3}.$

9. When $n=2$, $a=10$; when $n=6$, $a=-6$.

10. $a=-\frac{5}{3},\ l=\frac{43}{3}.$

11. 192 feet.

12. 400 feet, 1,024 feet, $16t^2$ feet.

Exercise 208

1. 384.
2. 405.
3. $l = -512$.
4. $a = \frac{4}{3}$.
5. $r = 2$.
6. 3, $\pm$ 6, 12, $\pm$ 24, 48.

7. The logarithms of the terms of a geometric series are $\log a$, $(\log a + \log r)$, $(\log a + 2\log r)$, $(\log a + 3\log r)$, $\cdots$ which is an arithmetic series with $\log a$ for its first term and $\log r$ for its difference.

8. $\text{Log}\, l = \log a + (n-1)\log r$.

   Solving for n, gives $n = \frac{\log l - \log a}{\log r} + 1$.

9. $\frac{y^2}{x}$.
10. $2 \cdot 3^{17} = 258 \cdot 10^6$.
11. .012175.
12. \$2.01.
13. \$131.80.
14. \$159.2

Exercise 209

1. $a = \frac{sr - s}{r^n - 1}$.
2. $a = rl + s - rs$, $r = \frac{s-a}{s-l}$, $l = \frac{sr - s + a}{r}$.
3. $s = 255$.
4. $s = 363$.
5. $s = 39 + 39\sqrt{3}$.
6. $s = 86$.
7. $s = \frac{211}{27}$.
8. $s = 73\frac{8}{9}$.
9. $\frac{x^7 - 1}{x - 1}$.
10. $\frac{x^5 + 1}{x + 1}$.
11. \$40.95.
12. 88,572.

Exercise 210

2. 9.

3. $\frac{9}{5}$.

4. 81.

5. $\frac{1}{3}$.

6. $12 + 6\sqrt{2}$.

7. $\frac{4(6 + 2\sqrt{3})}{3}$.

8. $\frac{1}{1 - x}$.

10. $\frac{7}{11}$.

12. $8\frac{1}{6}$.

13. $5\frac{1}{30}$.

14. $\frac{9}{37}$.

15. $\frac{160}{111}$.

16. 180 feet.

17. 60 in.

Exercise 211

1. Yes.

2. $\frac{a}{n}, \frac{a}{n}r, \frac{a}{n}r^2, \cdots$ is a G. P.

3. 100500.

4. 122.

5. $-7\frac{1}{2}, -2\frac{1}{2}, 2\frac{1}{2}, 7\frac{1}{2}$.

6. $s = \frac{2n+1}{2}(a + l)$ which is divisible by $2n + 1$.

7. $s = \frac{n}{2}[2 + (n - 1)2] = n^2$.

8. 3267.

9. 8.

10. $r = \pm 2$.

11. 3906.

12. $d = 3$.

13. $25\sqrt[3]{5}$.

14. 363.

15. 12, -6, and $+3$.

16. 5, 10, and 15.

17. $1,216.

18. 220.

19. Let $a = a$, $ar = b$, and $ar^2 = c$.

Then $\frac{1}{b - a} = \frac{1}{ar - a}$, $\frac{1}{2b} = \frac{1}{2ar}$, and

$$\frac{1}{b - c} = \frac{1}{ar - ar^2}.$$

Now $\frac{1}{ar - a}$, $\frac{1}{2ar}$, and $\frac{1}{ar - ar^2}$, given L. C. D.,

become $\frac{2r}{2ar(r - 1)}$, $\frac{r - 1}{2ar(r - 1)}$, and

$\frac{-2}{2ar(r - 1)}$ which is an A. P.

20. Second plan by \$12.50.
21. $39\frac{1}{4}$ miles.
22. 23400 feet.
23. 2900 feet.
24. 578.88 feet, 2315.52 feet.
26. 15 seconds.
27. 39.4 feet.
28. 6 seconds.
29. 72 inches.
30. $\left(\frac{9}{10}\right)^{10}$.
31. Twenty digits. $1837\cdot 10^{16}$.

Exercise 212

1. $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$.
2. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.
3. $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$.
4. $16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$.
5. $243a^5 - 810a^4b + 1080a^3b^2 - 720a^2b^3 + 240ab^4 - 32b^5$.
6. $32n^{10} + 240n^8m^2 + 720n^6m^4 + 1080n^4m^6 + 810n^2m^8 + 243m^{10}$.
7. $81x^8 - 540x^6y + 1350x^4y^2 - 1500x^2y^3 + 625y^4$.
8. $343R^6 - 882R^4r + 756R^2r^2 - 216r^3$.
9. $243m^{10}n^5 - 1620m^8n^4 + 4320m^6n^3 - 5760m^4n^2 + 3840m^2n - 1024$.
10. $1296x^4y^{12} - 1728a^2bx^3y^9 + 864a^4b^2x^2y^6 - 192a^6b^3xy^3 + 16a^8b^4$.
11. $16x^4 + 16x^3y + 6x^2y^2 + xy^3 + \frac{y^4}{16}$.
12. $\frac{x^3}{y^3} + \frac{3mx^2}{ny^2} + \frac{3m^2x}{n^2y} + \frac{m^3}{n^3}$.
13. $\frac{x^8}{y^8} - \frac{4x^4}{y^4} + 6 - \frac{4y^4}{x^4} + \frac{y^8}{x^8}$.

Exercise 213

1. $462a^5b^6$.
2. $-792x^5y^7$.
3. $4032a^5b^4$.
4. $22680n^4m^3$.
5. $240a^4b$.
6. $-\frac{189a^4b^5}{32}$.
7. $-\frac{5}{54}x^3y^3$.

Exercise 214

1. $a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 \cdots\cdot$.
2. $1 - x + x^2 - x^3 \cdots\cdot$.
3. $1 + 2x + 3x^2 + 4x^3 \cdots$.
4. $1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 \cdots\cdot$
5. $1 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{5}{16}x^3 \cdots\cdot$.
6. $16x^4 - \frac{16}{3}xy + \frac{2y^2}{9x^2} + \frac{y^3}{81x^5} \cdots\cdot$.
7. $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} \cdots\cdot$
8. $\frac{1}{x^3} + \frac{6y}{x^4} + \frac{24y^2}{x^5} + \frac{80y^3}{x^6} \cdots\cdot$.
9. $2^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{5}{2^{\frac{7}{2}}x^{\frac{3}{2}}} - \frac{75}{2^{\frac{17}{2}}x^{\frac{7}{2}}} - \frac{875}{2^{\frac{25}{2}}x^{\frac{11}{2}}} \cdots\cdot$.
10. $1024x^{10} + 5120x^9y + 11520x^8y^2 + 15360x^7y^3 \cdots$.
11. $3125x^{10} + 9375x^8b + 11250x^6b^2 + 6750x^4b^3 \cdots\cdot$.
12. $-8064x^5y^5$.
13. $7185024x^5y^6$.
14. $32x^5$.
15. $-\frac{5}{128}x^4$.
16. $\frac{429x^7}{2^{\frac{37}{2}}}$.
17. $1 + x - \frac{1}{2}x^2 \cdots\cdot$.

Exercise 216

1. $(x-2)(x+2)(x-3)$.
2. $(x-5)(x+2)(x+1)$.
3. $(a-3-5b)(a-3+5b)$.
4. $(a+1)(2a-3)(a-3)$.
5. $(a^2+ab+b^2)(a^2-ab+b^2)$.
6. $(x^2+2x+3)(x^2-2x+3)$.
7. $4a^2-2ab+b^2$.
8. 0.
9. $\frac{3x+4a}{x-a}$.
10. $\frac{m^2n^2(m^2+mn+n^2)}{ab(m+n)(a+b)}$.
11. $\frac{x(x^2+2xy+2y^2)}{(x+y)^2}$.
12. $\frac{a^3+a^2+a}{a^2+1}$.
13. $\frac{x^3-2x^2-x+1}{x^2-x-1}$.
15. $\frac{28}{37}$.
16. $\frac{x-1}{x}$.
17. $\frac{a^2-3a+1}{a^2-4a+1}$.
18. $\frac{m^2-m+1}{2m-1}$.

Exercise 217

1. $\frac{1}{16}$.
2. $\frac{1}{64}$.
3. 243.
4. 6.
5. $\frac{28}{3}$.
6. 972.
7. $\frac{1}{3}$.
8. $a=-2$.
9. $a=1$.
10. $a=1$, $a=-\frac{1}{4}$.
11. $a=2$.
12. $a=\frac{\sqrt{2}}{32}$.
13. $a=\frac{1}{625}$.
14. $a=\frac{5}{6}$.
15. $a=\frac{1}{32}$.
16. x^{7-2a}.
17. x.
18. $x^{a^3-b^3-a+b}$.
19. $x^{\frac{1}{n\pm1}}$.
20. $\frac{a^{\frac{1}{2}}}{x^{\frac{11}{48}}}$.
21. $x^{\frac{3a^2-2}{6a-2}}$.
22. $\frac{a^{\frac{5}{6}}\cdot x^{\frac{11}{12}}}{y^{\frac{1}{2}}}$.
23. $x^{\frac{1}{9}}$.
24. $a^{19}b^3c^9$.
25. $x^{\frac{a-b}{a+b}}$.

26. x^{2abc}.

27. $a^y b^x$.

28. $a^{2x^2-xy-y^2}$.

29. $\dfrac{x^{\frac{3}{c}}z^{12}}{y^{\frac{4}{a}}}$.

30. $\frac{5}{64}$.

31. $9\sqrt{3}$.

32. a^{4x+9}.

33. $\dfrac{20}{29}$.

34. $\dfrac{a^2x^2}{a^2-3ax+9x^2}$.

35. $\dfrac{1+a+a^2}{a^2}$.

36. .3.

37. 1.

38. $\dfrac{1}{4096}$.

39. $a^{3x}-3a^x+3a^{-x}-a^{-3x}$.

40. 59.

41. $4+16\sqrt{2}+3\sqrt{30}+\dfrac{34}{3}\sqrt{15}$.

42. $\sqrt{a^2-1}-a+1$.

43. $(a+13)\sqrt{a+1}-6a-14$.

44. $\dfrac{52-7\sqrt{10}}{1476}$.

45. $\dfrac{(3-\sqrt{2})\sqrt{2+\sqrt{3}}\sqrt{7}}{7}$.

46. $\dfrac{\sqrt{21}-3+2\sqrt{3}}{6}$.

47. $\dfrac{(2+\sqrt{3}-\sqrt{5}-\sqrt{2})(2\sqrt{3}+\sqrt{10})}{4}$.

48. $\dfrac{(3+\sqrt{2})(\sqrt{3+\sqrt{2}}-\sqrt{3})\sqrt{2}}{2}$.

49. $9-2\sqrt{6}$.

50. $\dfrac{(\sqrt{13+4\sqrt{3}})(\sqrt{3}+1)}{2}$.

51. $\frac{2}{3}\sqrt[5]{27a^2b^3}$.

52. $-\dfrac{31}{3}\sqrt{3}$.

53. $\dfrac{1+\sqrt{-2}}{3}$.

54. $\dfrac{1+2\sqrt{-2}}{3}$.

55. $-\dfrac{\sqrt{-1}}{a}$.

56. $\frac{\sqrt{3}+\sqrt{-2}+\sqrt{-3}-\sqrt{2}}{2}$. 57. $5-2\sqrt{6}$.

58. $\frac{x^2-x\sqrt{y-z}-x\sqrt{z-y}+(y-z)\sqrt{-1}}{x^2+y-z}$.

59. $-\frac{1+18\sqrt{-1}}{13}$. 60. $\frac{6m^2-mn\sqrt{-1}+2n^2}{4m^2+n^2}$.

61. $a^{-2x}+2a^{-x}-3$. 62. $2a^{\frac{3}{2}}-3a-a^{\frac{1}{2}}+2$.

63. $a^{\frac{5}{2}}-a^{\frac{3}{2}}x^{-1}+2a^{\frac{1}{2}}x^{-2}-a^{-\frac{1}{2}}x^{-3}$.

64. $a^3-\frac{1}{4}a^2+a-\frac{1}{2b}$.

65. .731. 66. .97. 67. $\frac{22}{49}$. 68. $\frac{a}{2}$.

69. $\frac{2a}{b}-\frac{2}{ab}\sqrt{a^4-b^4}$. 70. $\frac{(a+b)\sqrt{ab}}{2ab}$.

Exercise 218

1. $a=-2$.
2. $n=-5$.
3. $x=1,\ x=\frac{1}{4}$.
4. $a=-2$.
5. $x=4$.
6. $x=1,\ x=\frac{-3\pm\sqrt{5}}{2}$.
7. $a=2,\ a=-\frac{1}{3},\ a=\frac{5\pm\sqrt{37}}{6}$.
8. $x=4$.
9. $x=\frac{1}{2},\ x=5$.
10. $x=3$.
11. $x=-\frac{1}{2}$.
12. $x=-3,\ x=-\frac{7}{9}$.
13. $x=2,\ x=\frac{3}{2}$.
14. $x=2$.
15. $x=\frac{3}{2}$.
16. No root.

17. $x = 2,\ x = \frac{2}{3}.$

19. $x = \frac{5}{3},\ x = \frac{3}{2}.$

21. $x = 5.$

22. $x = -\frac{7}{4}.$

23. $x = 3,\ x = 4.$

24. $x = 1.$

25. $x = 4,\ x = -1.$

26. $x = 25.$

27. $x = \frac{1}{4}.$

28. $a = 8,\ a = -\frac{343}{125}.$

29. $a = 8,\ a = \frac{1}{8}.$

30. $x = \frac{b}{a}\quad x = -\frac{3b}{2a}.$

31. $x = \frac{1}{a}\quad x = -\frac{2}{b}.$

32. $x = \frac{2}{n},\ x = -\frac{3}{a}.$

33. $x = m - n,\ x = \frac{1}{m}.$

34. $x = \frac{6}{a},\ x = -\frac{1}{a},\ x = \frac{3}{a},\ x = -\frac{2}{a}.$

Exercise 219

1. $i = \frac{art}{rt + 1}.$ **2.** $C = \frac{5F - 160}{9}.$ **3.** $V = \frac{S}{6\pi}\sqrt{\pi S}.$

4. $l = a + \left[\frac{2s}{a + l} - 1\right]d.$

5. $n = \frac{\log l - \log a}{\log r} + 1.$

6. $s = \frac{l(1 - r^n)}{r^{n-1}(1 - r)}.$

7. $t = \frac{-v \pm \sqrt{v^2 + 2gs}}{g}.$

8. $v = \sqrt{2gs}.$

9. $x = 6.950,\ x = -1.689.$

10. 50.71.

11. 3.1416.

12. .5354.

13. $(a - b)(a^2 + ab + b^2)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$
$a^8 - a^7b + a^5b^3 - a^4b^4 + a^3b^5 - ab^7 + b^8).$

14. $x = 2,\ y = 3.$ **15.** $x = 7,\ y = -5.$ **16.** 7, 9, 11, 13.

17. 5 nickels, 7 dimes, and 8 quarters.

18. Rate of boat in still water 8 miles per hour, rate of stream 4 miles per hour.

19. 36.

20. 50 lbs. from the first, 40 lbs. from the second.

21. 50 cents per hour. **22.** 2160 feet. **23.** 12, 9.

24. Rates 12 and 18, distance 120 miles.

25. \$9,000 at 3%, \$8,000 at 4%. **26.** 27, 48.

27. A in 6, B in 12, C in 8 days.

28. 8 years.

29. $\frac{11}{14}$.

30. 7.416.

31. Apothem = 11.412 inches.
Area = 423.16 square inches.

32. $7\frac{1}{2}$ seconds.

33. 5.89 seconds.

35. At 11 A. M. and 4 P. M.

36. \$50. **37.** \$60. **38.** 12, 16, and 20 feet.

39. Of wagon wheels, 12 and 15 feet; of truck wheels, 10 and 12 feet.

40. At $\frac{64}{2 \cdots}$, by .0361; at $\frac{100}{2 \cdots}$, .00622.

41. Widths of fields 19.033 rods, 19.635 rods, 20.298 rods, and 21.034 rods.

42. 85.323 square feet.

43. 198.45 square feet.

44. $x = .618$.

45. $BD = 3.1415r$.

Exercise 222

1. $x = 3,\ x = \frac{-3 \pm 3\sqrt{-3}}{2}$.

2. $x = 2,\ x = -2,\ x = \pm 2\sqrt{-1}$.

3. $x = -2,\ x = 1 \pm \sqrt{-3}$.

4. $x = 3,\ x = \frac{-3 \pm 3\sqrt{-3}}{2},\ x = -3,\ x = \frac{3 \pm 3\sqrt{-3}}{2}$.

5. $x = -\frac{3}{2}$, $x = \frac{3 \pm 3\sqrt{-3}}{4}$.

6. $x = \pm 2\sqrt{-1}$, $x = 3 \pm \sqrt{-1}$, $x = -3 \pm \sqrt{-1}$.

Exercise 223

2. ∞.	**8.** 0.	**13.** 4.
4. 0.	**9.** 2.	**15.** 9.
5. 0.	**10.** 0.	**16.** 1.
6. 0.	**11.** ∞.	**17.** 11.
7. ∞.	**12.** ∞.	**18.** 0.

Exercise 224

1. $\sin A = \frac{a}{c} = \cos B$.

2. The hypotenuse is the longest side of a right triangle $\therefore$ both sine and cosine of an acute triangle are less than 1. The tangent may have any value.

3. The greater angle will have the greater sine and the greater tangent but the smaller cosine.

4. 45°. $\tan 45° = 1$.

9. $A = 52°$. $c = 21.57$. $b = 13.28$.

10. $A = 60°$, $B = 30°$, $a = 31.176$.

11. $B = 67°$, $a = 29.326$, $b = 38.975$.

12. $A = 47°$, $a = 6.68$, $c = 9.13$.

13. 31.043 feet.

14. 327.1 feet.

15. 172.5 feet.

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econd + the square of the
ast.

$(a^3 + b^3) = (a+b)(a^2 - ab + b^2)$

$(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

Table of Imaginary Unit

$(\sqrt{-1}) = i$

$(\sqrt{-1})^2 = i^2 = -1$

$(\sqrt{-1})^3 = i^3 = -\sqrt{-1} = -i$

$(\sqrt{-1})^4 = i^4 = 1$

$(\sqrt{-1})^5 = i^5 = i$

$(\sqrt{-1})^6 = i^6 = i^2 = -1$

$(\sqrt{-1})^7 = i^7 = -i$

$(\sqrt{-1})^8 = i^8 = -i^2 = 1$

$(\sqrt{-1})^9 = i^9 = i$

$(\sqrt{-1})^{10} = i^{10} = i^2 = -1$